

Superanalysis on quantum spaces

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Abstract

Attention is focused on antisymmetrised versions of quantum spaces that are of particular importance in physics, i.e. Manin plane, q -deformed Euclidean space in three or four dimensions as well as q -deformed Minkowski space. For each of these quantum spaces we provide q -analogs for elements of superanalysis, i.e. Grassmann integrals, Grassmann exponentials, Grassmann translations and braided products with supernumbers.

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1 Introduction

General motivation for deforming spacetime

Relativistic quantum field theory is not a fundamental theory, since its formalism leads to divergencies. In some cases like that of quantum electrodynamics one is able to overcome the difficulties with the divergencies by applying the so-called renormalization procedure due to Richard Feynman. Unfortunately, this procedure is not successful if we want to deal with quantum gravity. Despite the fact that gravitation is a rather weak interaction we are not able to treat it perturbatively. The reason for this lies in the fact, that transition amplitudes of n th order to the gravitation constant diverge like a momentum integral of the general form [1]

$$\int p^{2n-1} dp, \quad (1)$$

leaving us with an infinite number of ultraviolet divergent Feynman diagrams that cannot be removed by redefining finitely many physical parameters.

It is surely legitimate to ask for the reason for these fundamental difficulties. It is commonplace that the problems with the divergences in relativistic quantum field theory result from an incomplete description of spacetime at very small distances [2]. Niels Bohr and Werner Heisenberg have been the first who suggested that quantum field theories should be formulated on a spacetime lattice [3,4]. Such a spacetime lattice would imply the existence of a smallest distance a with the consequence that plane-waves of wave-length smaller than twice the lattice spacing could not propagate. In accordance with the relationship between wave-length λ and momentum p of a plane-wave, i.e.

$$\lambda \geq \lambda_{\min} = 2a \quad \Rightarrow \quad \frac{1}{\lambda} \sim p \leq p_{\max} \sim \frac{1}{2a}, \quad (2)$$

it follows then that physical momentum space would be bounded. Hence, the domain of all momentum integrals in Eq. (1) would be bounded as well with the consequence that momentum integrals should take on finite values.

q-Deformation of symmetries as an attempt to get a more detailed description of nature

Discrete spacetime structures in general do not respect classical Poincaré symmetry. A possible way out of this difficulty is to modify not only spacetime but also its corresponding symmetries. How are we to accomplish this? First of all let us recall that classical spacetime symmetries are usually described by Lie groups. Realizing that Lie groups are manifolds the

Gelfand-Naimark theorem tells us that Lie groups can be naturally embedded in the category of algebras [5]. The utility of this interrelation lies in formulating the geometrical structure of Lie groups in terms of a Hopf structure [6]. The point is that during the last two decades generic methods have been discovered for continuously deforming matrix groups and Lie algebras within the category of Hopf algebras. It is this development which finally led to the arrival of quantum groups and quantum spaces [7–13].

From a physical point of view the most realistic and interesting deformations are given by q -deformed versions of Minkowski space and Euclidean spaces as well as their corresponding symmetries, i.e. respectively Lorentz symmetry and rotational symmetry [14–18]. Further studies even allowed to establish differential calculi on these q -deformed quantum spaces [19–21] representing nothing other than q -analogs of classical translational symmetry. In this sense we can say that q -deformations of the complete Euclidean and Poincaré symmetries are now available [22]. Finally, Julius Wess and his coworkers were able to show that q -deformation of spaces and symmetries can indeed lead to the wanted discretizations of the spectra of spacetime observables [23,24], which nourishes the hope that q -deformation might give a new method to regularize quantum field theories [25–28].

Foundations of q -deformed superanalysis

In order to formulate quantum field theories on q -deformed quantum spaces it is necessary to provide us with some essential tools of a q -deformed analysis. The main question is how to define these new tools, which should be q -analogs of classical notions. Towards this end the considerations of Shahn Majid have proved very useful [29–31]. The key idea of his approach is that all the quantum spaces to a given quantum symmetry form a braided tensor category. Consequently, operations and objects concerning quantum spaces must rely on this framework of a braided tensor category, in order to guarantee their well-defined behavior under quantum group transformations. This so-called principle of covariance can be seen as the essential guideline for constructing a consistent theory.

In our previous work we have worked on symmetrized versions of quantum spaces that are of particular importance in physics, i.e. Manin plane, q -deformed Euclidean space with three or four dimensions as well as q -deformed Minkowski space. In each case we have presented explicit formulae for star-products [32], representations of symmetry generators and partial derivatives [33], q -integrals [34], q -exponentials [35] and q -translations [36]. But physics requires also antisymmetrized spaces, i.e. Grassmann algebras,

since they constitute an important tool in formulating supersymmetrical quantum field theories. In Ref. [37] we started showing that our ideas for symmetrized quantum spaces carry over to antisymmetrized ones as well.

Our goal now is to continue that program by providing explicit formulae for q-analogs of Grassmann integrals, Grassmann exponentials and Grassmann translations. In addition to this we are going to present formulae for braided products with supernumbers telling us how antisymmetrized quantum spaces have to be fused together with other quantum spaces.

The paper is organized as follows. In Sec. 2 we give a review of the concepts q-deformed superanalysis is based on. Furthermore, we recall some important results of Ref. [37]. This is done to an extent necessary for our further studies. In Sec. 3 we explain in detail how our general considerations apply to Manin plane. In Secs. 4 and 5 we repeat the same steps as in Sec. 3 for q-deformed Euclidean spaces in three and four dimensions, respectively. Sec. 6 is devoted to superanalysis on an antisymmetrized version of q-deformed Minkowski. Finally, in Sec. 7 we give a short conclusion and provide the reader with some interesting remarks about our new objects.

2 Concepts of q-deformed superanalysis

As already mentioned, q-deformed superanalysis is formulated within the framework of antisymmetrized quantum spaces. These quantum spaces are defined as modules of quasitriangular Hopf algebras which describe the underlying symmetry. For our purposes, it is at first sufficient to consider an antisymmetrized quantum space as an algebra generated by coordinates $\theta^1, \theta^2, \dots, \theta^n$ which are subjected to

$$\theta^i \theta^j = -k(\hat{R})^{ij}_{kl} \theta^k \theta^l, \quad k \in \mathbb{R}^+, \quad (3)$$

where \hat{R} denotes a representation of the universal \mathcal{R} -matrix assigned to the underlying quantum symmetry. This way, we get nothing other than q-deformed versions of Grassman algebras.

Moreover, it is important to realize that our antisymmetrized quantum spaces satisfy the so-called *Poincaré-Birkhoff-Witt property*, i.e. the dimension of a subspace of homogenous polynomials should be the same as for classical Grassmann variables. This property is the deeper reason why normal ordered monomials constitute a basis of our q-deformed Grassmann algebras. Consequently, each supernumber can be represented in the general form

$$f(\underline{\theta}) = f' + \sum_{\underline{K}} f_{\underline{K}} \theta^{\underline{K}}, \quad (4)$$

where θ^K denotes monomials of a given normal ordering.

For this to become more clear, the two-dimensional antisymmetrized quantum plane shall serve as an example [38]. By specifying the R-matrix in Eq. (3) to that of $U_q(su_2)$, we obtain as defining relations of antisymmetrized Manin plane

$$\begin{aligned}(\theta^1)^2 &= (\theta^2)^2 = 0, \\ \theta^1\theta^2 &= -q^{-1}\theta^2\theta^1,\end{aligned}\tag{5}$$

showing the correct classical limit for $q \rightarrow 1$. Due to these relations, each supernumber can be written in the general form (notice that the normal ordering of monomials is indicated by the order in which coordinates are arranged in the symbol for supernumbers)

$$f(\theta^2, \theta^1) = f' + f_1\theta^1 + f_2\theta^2 + f_{21}\theta^2\theta^1,\tag{6}$$

and the product of two such supernumbers finally becomes

$$\begin{aligned}(f \cdot g)(\theta^2, \theta^1) \\ = (f \cdot g)' + (f \cdot g)_1\theta^1 + (f \cdot g)_2\theta^2 + (f \cdot g)_{21}\theta^2\theta^1,\end{aligned}\tag{7}$$

with

$$\begin{aligned}(f \cdot g)' &= f'g', \\ (f \cdot g)_\alpha &= f_\alpha g' + f'g_\alpha, \quad \alpha = 1, 2, \\ (f \cdot g)_{21} &= f_2g_1 - q^{-1}f_1g_2.\end{aligned}\tag{8}$$

Similar results hold for the other antisymmetrized quantum spaces we consider in this article [37].

Next, we would like to come to the covariant differential calculi on our antisymmetrized quantum spaces [19, 20, 39]. In complete accordance to symmetrized quantum spaces, there exist always two covariant differential calculi. Their Leibniz rules take the general form

$$\begin{aligned}\partial_\theta^i \theta^j &= g^{ij} - k(\hat{R}^{-1})^{ij}_{kl} \theta^k \partial_\theta^l, \quad k \in \mathbb{C}, \\ \hat{\partial}_\theta^i \theta^j &= g^{ij} - k^{-1}(\hat{R})^{ij}_{kl} \theta^k \hat{\partial}_\theta^l,\end{aligned}\tag{9}$$

where g^{ij} denotes the in the corresponding quantum metric (as reference, we provide a review of key notations in Appendix A). In the two-dimensional case, for example, the relations for the first differential calculus read explicitly

$$\partial_\theta^1 \theta^1 = -q^{-1}\theta^1 \partial_\theta^1,\tag{10}$$

$$\begin{aligned}
\partial_\theta^1 \theta^2 &= -q^{-1/2} - \theta^2 \partial_\theta^1, \\
\partial_\theta^2 \theta^1 &= q^{1/2} - \theta^1 \partial_\theta^2 + \lambda \theta^2 \partial_\theta^1, \\
\partial_\theta^2 \theta^2 &= -q^{-1} \theta^2 \partial_\theta^2,
\end{aligned} \tag{11}$$

leading to the following actions on supernumbers [37]:

$$\begin{aligned}
(\partial_\theta)_1 \triangleright f(\theta^2, \theta^1) &= f_1 - q^{-1} f_{21} \theta^2, \\
(\partial_\theta)_2 \triangleright f(\theta^2, \theta^1) &= f_2 + f_{21} \theta^1.
\end{aligned} \tag{12}$$

However, in what follows it is necessary to take another point of view which is provided by category theory. A category is a collection of objects X, Y, Z, \dots together with a set $\text{Mor}(X, Y)$ of morphisms between two objects X, Y . The composition of morphisms has similar properties as the composition of maps. We are interested in tensor categories. These categories have a product, denoted \otimes and called the tensor product. It admits several 'natural' properties such as associativity and existence of a unit object. For a more formal treatment we refer to Refs. [29, 30], [40] or [41]. If the action of a quasitriangular Hopf algebra \mathcal{H} on the tensor product of two quantum spaces X and Y is defined by

$$h \triangleright (v \otimes w) = (h_{(1)} \triangleright v) \otimes (h_{(2)} \triangleright w) \in X \otimes Y, \quad h \in \mathcal{H}, \tag{13}$$

where the coproduct is written in the so-called Sweedler notation, i.e. $\Delta(h) = h_{(1)} \otimes h_{(2)}$, then the representations (quantum spaces) of the given Hopf algebra (quantum algebra) are the objects of a tensor category.

In this tensor category exist a number of morphisms of particular importance that are covariant with respect to the Hopf algebra action. First of all, for any pair of objects X, Y there is an isomorphism $\Psi_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ such that $(g \otimes f) \circ \Psi_{X,Y} = \Psi_{X',Y'} \circ (f \otimes g)$ for arbitrary morphisms $f \in \text{Mor}(X, X')$ and $g \in \text{Mor}(Y, Y')$. In addition to this one requires the hexagon axiom to hold. The hexagon axiom is the validity of the two conditions

$$\Psi_{X,Y} \circ \Psi_{Y,Z} = \Psi_{X \otimes Y, Z}, \quad \Psi_{X,Z} \circ \Psi_{X,Y} = \Psi_{X, Y \otimes Z}. \tag{14}$$

A tensor category equipped with such mappings $\Psi_{X,Y}$ for each pair of objects X, Y is called a braided tensor category. The mappings $\Psi_{X,Y}$ as a whole are often referred to as the braiding of the tensor category. Furthermore, for any quantum space algebra X in this category there are morphisms $\Delta : X \rightarrow X \otimes X$, $S : X \rightarrow X$ and $\varepsilon : X \rightarrow \mathbb{C}$ forming a braided Hopf algebra, i.e. Δ , S and ε obey the usual axioms of a Hopf algebra, but now

as morphisms in the braided category. For further details we recommend Refs. [40] and [42].

The explicit form of these morphisms is completely determined by the so-called \mathcal{L} -matrix [12, 22, 43]. The entries of the \mathcal{L} -matrix are built up out of symmetry generators and scaling operators. For the quantum spaces we study in this article, the explicit form of the \mathcal{L} -matrix can be looked up in Ref. [37]. To be more concrete, we give as example the non-vanishing entries of the \mathcal{L} -matrix and its conjugate in the case of Manin plane:

$$\begin{aligned}(\mathcal{L}_a)_1^1 &= \Lambda(a)\tau^{-1/4}, \\ (\mathcal{L}_a)_1^2 &= -q\lambda\Lambda(a)\tau^{-1/4}T^+, \\ (\mathcal{L}_a)_2^2 &= \Lambda(a)\tau^{1/4},\end{aligned}\tag{15}$$

and likewise

$$\begin{aligned}(\bar{\mathcal{L}}_a)_1^1 &= \Lambda^{-1}(a)\tau^{1/4}, \\ (\bar{\mathcal{L}}_a)_2^1 &= -q^{-1}\lambda\Lambda^{-1}(a)\tau^{-1/4}T^-, \\ (\bar{\mathcal{L}}_a)_2^2 &= \Lambda^{-1}(a)\tau^{1/4},\end{aligned}\tag{16}$$

where $\tau^{\pm 1/4}$, T^{\pm} and $\Lambda(a)$ denote generators of the quantum algebra $U_q(su_2)$ and a unitary scaling operator, respectively.

Using the \mathcal{L} -matrix and its conjugate, the two distinct braidings of a quantum space generator a^i can be obtained in the compact form

$$\begin{aligned}\Psi_{X,Y}(a^i \otimes f) &= ((\bar{\mathcal{L}}_a)_j^i \triangleright f) \otimes a^j, \\ \Psi_{X,Y}^{-1}(a^i \otimes f) &= ((\mathcal{L}_a)_j^i \triangleright f) \otimes a^j\end{aligned}\tag{17}$$

$$\begin{aligned}\Psi_{X,Y}(f \otimes a^i) &= a^j \otimes (f \triangleleft (\mathcal{L}_a)_j^i), \\ \Psi_{X,Y}^{-1}(f \otimes a^i) &= a^j \otimes (f \triangleleft (\bar{\mathcal{L}}_a)_j^i).\end{aligned}\tag{18}$$

There remains to evaluate the action of the \mathcal{L} -matrix on the quantum space element f . This can be achieved by making use of the representations presented in Refs. [33] and [37].

By repeated use of the identities in Eq. (17) we are able to calculate the braiding between a monomial in Grassmann variables θ^i and an arbitrary element g of another quantum space, i.e.

$$\begin{aligned}(\theta^i \dots \theta^j) \underline{\odot}_{\bar{\mathcal{L}}} g &\equiv \Psi(\theta^i \dots \theta^j \otimes g) \\ &= \left((\bar{\mathcal{L}}_{\theta})_{k_i}^i \dots (\bar{\mathcal{L}}_{\theta})_{k_j}^j \triangleright g \right) \otimes (\theta^{k_i} \dots \theta^{k_j}),\end{aligned}\tag{19}$$

$$\begin{aligned}
(\theta^i \dots \theta^j) \underline{\odot}_L g &\equiv \Psi^{-1}(\theta^i \dots \theta^j \otimes g) \\
&= \left((\mathcal{L}_\theta)_{k_i}^i \dots (\mathcal{L}_\theta)_{k_j}^j \triangleright g \right) \otimes (\theta^{k_i} \dots \theta^{k_j}), \\
g \underline{\odot}_R (\theta^i \dots \theta^j) &\equiv \Psi(g \otimes \theta^i \dots \theta^j) \\
&= (\theta^{k_i} \dots \theta^{k_j}) \otimes \left(g \triangleleft (\mathcal{L}_\theta)_{k_i}^i \dots (\mathcal{L}_\theta)_{k_j}^j \right), \\
g \underline{\odot}_{\bar{R}} (\theta^i \dots \theta^j) &\equiv \Psi^{-1}(g \otimes \theta^i \dots \theta^j) \\
&= (\theta^{k_i} \dots \theta^{k_j}) \otimes \left(g \triangleleft (\bar{\mathcal{L}}_\theta)_{k_i}^i \dots (\bar{\mathcal{L}}_\theta)_{k_j}^j \right).
\end{aligned} \tag{20}$$

Recalling that the braiding mappings are linear in their arguments, it should be quite clear that the braiding of a supernumber with an arbitrary element g is completely determined by the above identities. In the subsequent sections this observation will enable us to derive explicit formulae for the braiding of supernumbers with arbitrary quantum space elements. The resulting expressions are referred to as braided products for supernumbers.

Using \mathcal{L} -matrices, the coproducts for quantum space generators a^i can be obtained in the general form

$$\begin{aligned}
\Delta_{\bar{L}}(a^i) &= a^i \otimes 1 + (\bar{\mathcal{L}}_a)_j^i \otimes a^j, \\
\Delta_L(a^i) &= a^i \otimes 1 + (\mathcal{L}_a)_j^i \otimes a^j,
\end{aligned} \tag{21}$$

and the corresponding antipodes are then given by (if we assume for the counit $\varepsilon(\theta^i) = 0$)

$$\begin{aligned}
S_{\bar{L}}(\theta^i) &= -S(\bar{\mathcal{L}}_\theta)_j^i \theta^j, \\
S_L(\theta^i) &= -S(\mathcal{L}_\theta)_j^i \theta^j.
\end{aligned} \tag{22}$$

The essential observation for this paper is that coproducts of coordinates imply their translations [22, 44–46]. This can be seen as follows. The coproduct Δ on coordinates is an algebra homomorphism. If the coordinates constitute a module coalgebra then the algebra structure of the coordinates a^i is carried over to their coproduct $\Delta(a^i)$. More formally, we have

$$\Delta(a^i a^j) = \Delta(a^i) \Delta(a^j) \quad \text{and} \quad \Delta(h \triangleright a^i) = \Delta(h) \triangleright \Delta(a^i). \tag{23}$$

Due to this fact we can think of (21) as nothing other than an addition law for q -deformed vector components. In this article we use this fact to derive translation operations for q -deformed supernumbers.

Next, let us make contact with another important ingredient of q -deformed superanalysis, i.e. q -deformed Grassmann exponential. For this purpose we

have to suppose that our category is equipped with a dual object X^* for each algebra X in the category. This means that we have dual pairings

$$\langle \cdot, \cdot \rangle : X \otimes X^* \rightarrow \mathbb{C} \quad \text{with} \quad \langle e_a, f^b \rangle = \delta_a^b, \quad (24)$$

where $\{e_a\}$ is a basis in X and $\{f_a\}$ a dual basis in X^* . Now, we are able to introduce an exponential map [44] which is defined to be the dual object of (24). Thus, the exponential is given by

$$\exp : \mathbb{C} \rightarrow X^* \otimes X, \quad \text{with} \quad \exp = \sum_a f^a \otimes e_a. \quad (25)$$

It was shown in Ref. [31] that there is such a dual pairing of Grassmann variables and corresponding partial derivatives. Specifically, we have

$$\langle \cdot, \cdot \rangle : \mathcal{M}_\partial \otimes \mathcal{M}_\theta \rightarrow \mathbb{C} \quad \text{with} \quad \langle f(\underline{\partial}_\theta), g(\underline{\theta}) \rangle \equiv \varepsilon(f(\underline{\partial}_\theta) \triangleright g(\underline{\theta})). \quad (26)$$

If we know a basis of the coordinate algebra \mathcal{M}_θ being dual to a given one of \mathcal{M}_∂ , then we will be able to read off from the definition in Eq. (25) the explicit form of the q-exponentials. This task will be done in the subsequent sections for all quantum spaces under consideration.

3 Two-Dimensional quantum plane

In this section we present explicit formulae for elements of q-deformed superanalysis on antisymmetrized Manin plane (for its definition see also Appendix A). To begin with, we introduce the notion of a left-superintegral. With the partial derivatives obeying the relations in Eq. (10) this can be done in complete analogy to the undeformed case:

$$\int f(\theta^2, \theta^1) d_L^2 \theta \equiv (\partial_\theta)_1 (\partial_\theta)_2 \triangleright f(\theta^2, \theta^1) = f_{21}. \quad (27)$$

This integral shows the same properties as its classical counterpart. Thus, it is linear, normed and translationally invariant. Linearity is clear because of the linearity of the derivatives and the other two properties follow from its very definition, since we have

$$\int \theta^2 \theta^1 d_L^2 \theta = 1, \quad \int \theta^\alpha d_L^2 \theta = \int d_L^2 \theta = 0, \quad \alpha = 1, 2, \quad (28)$$

and

$$\int (\partial_\theta)_\alpha \triangleright f(\theta^2, \theta^1) d_L^2 \theta = 0, \quad \alpha = 1, 2. \quad (29)$$

We could also have started our considerations with the conjugated partial derivatives $(\hat{\partial}_\theta)_\alpha$ whose representations are linked to those in (12) via the correspondence (see Ref. [37])

$$(\hat{\partial}_\theta)_\alpha \bar{\triangleright} f(\theta^1, \theta^2) \xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} -(\partial_\theta)_{\alpha'} \triangleright f(\theta^2, \theta^1), \quad (30)$$

where $\alpha' = 3 - \alpha$. The symbol $\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'}$ denotes a transition between the two expressions via the substitutions

$$\begin{aligned} \theta^\alpha &\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} \theta^{\alpha'}, & \theta^\alpha \theta^\beta &\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} \theta^{\alpha'} \theta^{\beta'}, & q &\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} q^{-1}, \\ f' &\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} f', & f_\alpha &\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} f_{\alpha'}, & f_{\alpha\beta} &\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} f_{\alpha'\beta'}, \quad \alpha, \beta = 1, 2. \end{aligned} \quad (31)$$

The corresponding superintegral then becomes

$$\int f(\theta^1, \theta^2) d_L^2 \theta = (\hat{\partial}_\theta)_2 (\hat{\partial}_\theta)_1 \bar{\triangleright} f(\theta^1, \theta^2) = f_{12}. \quad (32)$$

Notice that in Eqs. (27) and (32) the subscripts at the integration measure help us to distinguish the two types of superintegrals. Using the action of conjugated partial derivatives on Grassmann variables, it is again straightforward to show that

$$\int \theta^1 \theta^2 d_L^2 \theta = 1, \quad \int \theta^\alpha d_L^2 \theta = \int d_L^2 \theta = 0, \quad \alpha = 1, 2, \quad (33)$$

and

$$\int (\partial_\theta)_\alpha \bar{\triangleright} f(\theta^1, \theta^2) d_L^2 \theta = 0, \quad \alpha = 1, 2. \quad (34)$$

However, superintegrals can also be constructed from partial derivatives acting on a supernumber from the right. Let us recall that left and right derivatives are related to each other by (see Ref. [37])

$$\begin{aligned} f(\theta^1, \theta^2) \triangleleft (\hat{\partial}_\theta)_\alpha &\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} -(\hat{\partial}_\theta)_{\alpha'} \bar{\triangleright} f(\theta^1, \theta^2), \\ f(\theta^2, \theta^1) \triangleleft (\partial_\theta)_\alpha &\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} -(\partial_\theta)_{\alpha'} \triangleright f(\theta^2, \theta^1), \end{aligned} \quad (35)$$

where $\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'}$ now stands for the transition

$$\theta^\alpha \xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} \theta^{\alpha'}, \quad \theta^\alpha \theta^\beta \xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} \theta^{\beta'} \theta^{\alpha'}, \quad (36)$$

$$f' \xleftrightarrow{\alpha \leftrightarrow \alpha'} f', \quad f^\alpha \xleftrightarrow{\alpha \leftrightarrow \alpha'} f^{\alpha'}, \quad f^{\alpha\beta} \xleftrightarrow{\alpha \leftrightarrow \alpha'} f^{\beta'\alpha'}.$$

Using right derivatives, we can proceed in very much the same way as was done for left derivatives. This way we introduce

$$\begin{aligned} \int d_R^2 \theta f(\theta^1, \theta^2) &\equiv f(\theta^1, \theta^2) \triangleleft (\hat{\partial}_\theta)_2 (\hat{\partial}_\theta)_1 = f_{12}, \\ \int d_R^2 \theta f(\theta^2, \theta^1) &= f(\theta^2, \theta^1) \bar{\triangleleft} (\partial_\theta)_1 (\partial_\theta)_2 = f_{21}. \end{aligned} \quad (37)$$

The new definitions lead immediately to

$$\begin{aligned} \int d_R^2 \theta \theta^1 \theta^2 &= 1, \quad \int d_R^2 \theta \theta^\alpha = \int d_R^2 \theta = 0, \\ \int d_R^2 \theta \theta^2 \theta^1 &= 1, \quad \int d_R^2 \theta \theta^\alpha = \int d_R^2 \theta = 0, \quad \alpha = 1, 2, \end{aligned} \quad (38)$$

and

$$\begin{aligned} \int d_R^2 \theta f(\theta^1, \theta^2) \triangleleft \hat{\partial}_\theta^\alpha &= 0, \\ \int d_R^2 \theta f(\theta^2, \theta^1) \bar{\triangleleft} \partial_\theta^\alpha &= 0, \quad \alpha = 1, 2. \end{aligned} \quad (39)$$

Next, we come to the q-deformed superexponential. For its calculation we need to know the dual pairing between partial derivatives and coordinates. Explicitly, we have as non-vanishing expressions

$$\langle (\partial_\theta)_2, \theta^2 \rangle_{L, \bar{R}} = \langle (\partial_\theta)_1, \theta^1 \rangle_{L, \bar{R}} = \langle (\partial_\theta)_1 (\partial_\theta)_2, \theta^2 \theta^1 \rangle_{L, \bar{R}} = 1, \quad (40)$$

which follow from the very definition of the dual pairing together with the action of partial derivatives on Grassmann variables. From the above result we can at once read off the elements of the two bases being dual to each other. Inserting these elements into the general formulae for the exponential in Eq. (25) we obtain as explicit form of the q-deformed superexponential on Manin plane

$$\exp(\theta_{\bar{R}} \mid (\partial_\theta)_L) = 1 \otimes 1 + \theta^1 \otimes (\partial_\theta)_1 + \theta^2 \otimes (\partial_\theta)_2 + \theta^2 \theta^1 \otimes (\partial_\theta)_1 (\partial_\theta)_2. \quad (41)$$

Repeating the same steps as before for the conjugated partial derivatives we get instead

$$\langle (\hat{\partial}_\theta)_1, \theta^1 \rangle_{\bar{L}, R} = \langle (\hat{\partial}_\theta)_2, \theta^2 \rangle_{\bar{L}, R} = \langle (\hat{\partial}_\theta)_2 (\hat{\partial}_\theta)_1, \theta^1 \theta^2 \rangle_{\bar{L}, R} = 1, \quad (42)$$

and consequently

$$\exp(\theta_R \mid (\hat{\partial}_\theta)_{\bar{L}}) = 1 \otimes 1 + \theta^1 \otimes (\hat{\partial}_\theta)_1 + \theta^2 \otimes (\hat{\partial}_\theta)_2 + \theta^1 \theta^2 \otimes (\hat{\partial}_\theta)_2 (\hat{\partial}_\theta)_1. \quad (43)$$

The above two results together with those corresponding to unconjugated partial derivatives establish the correspondences

$$\begin{aligned} \langle \underline{\partial}_\theta, \underline{\theta} \rangle_{L, \bar{R}} &\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} \langle \hat{\underline{\partial}}_\theta, \underline{\theta} \rangle_{\bar{L}, R}, \\ \exp(\theta_{\bar{R}} \mid (\partial_\theta)_L) &\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} \exp(\theta_R \mid (\hat{\partial}_\theta)_{\bar{L}}), \end{aligned} \quad (44)$$

where the symbol $\xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'}$ now denotes a transition via

$$\theta^\alpha \leftrightarrow \theta^{\alpha'}, \quad (\partial_\theta)_\alpha \leftrightarrow (\hat{\partial}_\theta)_{\alpha'}, \quad q \leftrightarrow q^{-1}. \quad (45)$$

The above considerations on dual pairings and superexponentials are based on the use of left derivatives, but they carry over to right derivatives as well with a few necessary modifications. Towards this end, we have to realize that the definitions

$$\begin{aligned} \langle f(\underline{\theta}), g(\underline{\partial}_\theta) \rangle_{L, \bar{R}} &\equiv \varepsilon(f(\underline{\theta}) \bar{\lhd} g(\underline{\partial}_\theta)), \\ \langle f(\underline{\theta}), g(\hat{\underline{\partial}}_\theta) \rangle_{\bar{L}, R} &\equiv \varepsilon(f(\underline{\theta}) \lhd g(\hat{\underline{\partial}}_\theta)), \end{aligned} \quad (46)$$

give a dual pairing as well. Now, we can repeat the same reasonings as above. This way we get

$$\begin{aligned} \langle \theta_1, \partial_\theta^1 \rangle_{L, \bar{R}} &= \langle \theta_2, \partial_\theta^2 \rangle_{L, \bar{R}} = \langle \theta_1 \theta_2, \partial_\theta^2 \partial_\theta^1 \rangle_{L, \bar{R}} = 1, \\ \langle \theta_1, \hat{\partial}_\theta^1 \rangle_{\bar{L}, R} &= \langle \theta_2, \hat{\partial}_\theta^2 \rangle_{\bar{L}, R} = \langle \theta_2 \theta_1, \hat{\partial}_\theta^1 \hat{\partial}_\theta^2 \rangle_{\bar{L}, R} = 1, \end{aligned} \quad (47)$$

and

$$\begin{aligned} \exp((\partial_\theta)_{\bar{R}} \mid \theta_L) &= 1 \otimes 1 + \partial_\theta^1 \otimes \theta_1 + \partial_\theta^2 \otimes \theta_2 + \partial^2 \partial^1 \otimes \theta_1 \theta_2, \\ \exp((\hat{\partial}_\theta)_R \mid \theta_{\bar{L}}) &= 1 \otimes 1 + \hat{\partial}_\theta^1 \otimes \theta_1 + \hat{\partial}_\theta^2 \otimes \theta_2 + \hat{\partial}^1 \hat{\partial}^2 \otimes \theta_2 \theta_1. \end{aligned} \quad (48)$$

Comparing these results to those for left derivatives shows us the existence of the crossing symmetries

$$\begin{aligned} \langle \underline{\theta}, \underline{\partial}_\theta \rangle_{L, \bar{R}} &\xleftrightarrow{\alpha \leftrightarrow \alpha'} \langle \underline{\partial}_\theta, \underline{\theta} \rangle_{L, \bar{R}}, \\ \langle \underline{\theta}, \hat{\underline{\partial}}_\theta \rangle_{\bar{L}, R} &\xleftrightarrow{\alpha \leftrightarrow \alpha'} \langle \hat{\underline{\partial}}_\theta, \underline{\theta} \rangle_{\bar{L}, R}, \end{aligned} \quad (49)$$

and

$$\begin{aligned} \exp((\partial_\theta)_{\bar{R}} | \theta_L) &\stackrel{\alpha \leftrightarrow \alpha'}{\longleftrightarrow} \exp(\theta_{\bar{R}} | (\partial_\theta)_L) , \\ \exp((\hat{\partial}_\theta)_R | \theta_{\bar{L}}) &\stackrel{\alpha \leftrightarrow \alpha'}{\longleftrightarrow} \exp(\theta_R | (\hat{\partial}_\theta)_{\bar{L}}) , \end{aligned} \quad (50)$$

where $\stackrel{\alpha \leftrightarrow \alpha'}{\longleftrightarrow}$ indicates one of the following two substitutions:

$$\begin{aligned} \text{a) } (\partial_\theta)_\alpha &\leftrightarrow \theta_\alpha, \quad \theta^\alpha \leftrightarrow \partial_\theta^\alpha , \\ \text{b) } (\hat{\partial}_\theta)_\alpha &\leftrightarrow \theta_\alpha, \quad \theta^\alpha \leftrightarrow \hat{\partial}_\theta^\alpha . \end{aligned} \quad (51)$$

Next, we would like to deal with Grassmann translations. As was pointed out in Sec. 2 translations on quantum spaces are described by the coproduct, which on spinor coordinates reads [37]

$$\begin{aligned} \Delta_{\bar{L}}(\theta^1) &= \theta^1 \otimes 1 + \tilde{\Lambda} \tau^{1/4} \otimes \theta^1 + q^{-1} \lambda \tilde{\Lambda} \tau^{-1/4} T^- \otimes \theta^2, \\ \Delta_{\bar{L}}(\theta^2) &= \theta^2 \otimes 1 + \tilde{\Lambda} \tau^{-1/4} \otimes \theta^2. \end{aligned} \quad (52)$$

Notice that τ , T^+ and $\tilde{\Lambda}$ denote generators of $U_q(su(2))$ and a scaling operator, respectively. Now, we can follow the same reasonings already applied in Ref. [36]. In this manner the coproduct is split into two parts by introducing left and right coordinates:

$$\Delta_{\bar{L}}(\theta^\alpha) = \theta^\alpha \otimes 1 + (\bar{\mathcal{L}}_\theta)_\beta^\alpha \otimes \theta^\beta = \theta_l^\alpha + \theta_r^\alpha, \quad (53)$$

where

$$\theta_l^\alpha \equiv \theta_l^\alpha \otimes 1, \quad \theta_r^\alpha \equiv (\bar{\mathcal{L}}_\theta)_\beta^\alpha \otimes \theta^\beta, \quad \alpha = 1, 2. \quad (54)$$

Since the entries of the \mathcal{L} -matrix are built up out of symmetry generators, the commutation relations between right and left coordinates can be derived in a straightforward manner from the commutation relations between symmetry generators and Grassmann variables (their explicit form was given in Ref. [37]). It follows that

$$\begin{aligned} \theta_r^1 \theta_l^1 &= -\theta_l^1 \theta_r^1, \\ \theta_r^1 \theta_l^2 &= -q^{-1} \theta_l^2 \theta_r^1 - q^{-1} \lambda \theta_l^1 \theta_r^2, \\ \theta_r^2 \theta_l^1 &= -q^{-1} \theta_l^1 \theta_r^2, \\ \theta_r^2 \theta_l^2 &= -\theta_l^2 \theta_r^2. \end{aligned} \quad (55)$$

Furthermore, these relations imply

$$\Delta_{\bar{L}}(\theta^2 \theta^1) = \Delta_{\bar{L}}(\theta^2) \Delta_{\bar{L}}(\theta^1) = (\theta_l^2 + \theta_r^2)(\theta_l^1 + \theta_r^1) \quad (56)$$

$$\begin{aligned}
&= \theta_l^2 \theta_l^1 + \theta_l^2 \theta_r^1 + \theta_r^2 \theta_l^1 + \theta_r^2 \theta_r^1 \\
&= \theta_l^2 \theta_l^1 + \theta_l^2 \theta_r^1 - q^{-1} \theta_l^1 \theta_r^2 + \theta_r^2 \theta_r^1.
\end{aligned}$$

Notice that in the last step we have switched all right coordinates to the right of all left coordinates. Now, we are in a position to read off from the above results the explicit form of a Grassmann translation. Specifically, it becomes for a supernumber written in the form of Eq. (6)

$$\begin{aligned}
f(\theta \oplus_{\bar{L}} \psi) &\equiv f(\theta_l^2 + \theta_r^2, \theta_l^2 + \theta_r^1) |_{\theta_l^\alpha \rightarrow \theta^\alpha, \theta_r^\alpha \rightarrow \Psi^\alpha} \\
&= [f' + f_1(\theta^1 + \psi^1) + f_2(\theta^2 + \psi^2) \\
&\quad + f_{21}(\theta_l^2 + \theta_r^2)(\theta_l^2 + \theta_r^1)] |_{\theta_l^\alpha \rightarrow \theta^\alpha, \theta_r^\alpha \rightarrow \Psi^\alpha} \\
&= f' + f_1(\theta^1 + \psi^1) + f_2(\theta^2 + \psi^2) \\
&\quad + f_{21}(\theta^2 \theta^1 + \theta^2 \psi^1 - q^{-1} \theta^1 \psi^2 + \psi^2 \psi^1).
\end{aligned} \tag{57}$$

For a proper understanding of the definition above, one should realize that substitutions can only be performed after all right coordinates have been commuted to the right of an expression.

In order to introduce translations in the opposite direction we need to consider the antipode corresponding to the coproducts in Eq. (52). On spinor coordinates this antipode takes the form [37]

$$\begin{aligned}
S_{\bar{L}}(\theta^1) &= -\tilde{\Lambda}^{-1} \tau^{-1/4} \theta^1 + q^{-2} \lambda \tilde{\Lambda}^{-1} \tau^{-1/4} T^- \theta^2, \\
S_{\bar{L}}(\theta^2) &= \tilde{\Lambda}^{-1} \tau^{1/4} \theta^2.
\end{aligned} \tag{58}$$

However, what we are looking for is an antipode in terms of right or left coordinates. To achieve this, we have to exploit the Hopf algebra axiom

$$m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta = \varepsilon, \tag{59}$$

where m denotes multiplication in the Grassmann algebra. If we substitute for the coproduct the expressions in Eqs. (53) and (56), we arrive at

$$S_{\bar{L}}(\theta^\alpha) + \theta^\alpha = \theta^\alpha + S_{\bar{L}}(\theta^\alpha) = 0, \quad \alpha = 1, 2, \tag{60}$$

and

$$\begin{aligned}
&S_{\bar{L}}(\theta^2 \theta^1) + S_{\bar{L}}(\theta^2) \theta^1 - q^{-1} S_{\bar{L}}(\theta^1) \theta^2 + \theta^2 \theta^1 \\
&= \theta^2 \theta^1 + \theta^2 S_{\bar{L}}(\theta^1) - q^{-1} \theta^1 S_{\bar{L}}(\theta^2) + S_{\bar{L}}(\theta^2 \theta^1) = 0.
\end{aligned} \tag{61}$$

This system of equations can be solved for $S_{\bar{L}}(\theta^\alpha)$, $\alpha = 1, 2$, and $S_{\bar{L}}(\theta^2\theta^1)$, leaving us with

$$\begin{aligned} S_{\bar{L}}(\theta^\alpha) &= -\theta^\alpha, \quad \alpha = 1, 2, \\ S_{\bar{L}}(\theta^2\theta^1) &= q^{-2}\theta^2\theta^1. \end{aligned} \quad (62)$$

Finally, the last results allow us to introduce the following operation on supernumbers:

$$\begin{aligned} f(\underline{\ominus}_{\bar{L}}\theta) &\equiv S_{\bar{L}}(f(\theta^2, \theta^1)) \\ &= f' + f_1 S_{\bar{L}}(\theta^1) + f_2 S_{\bar{L}}(\theta^2) + f_{21} S_{\bar{L}}(\theta^2\theta^1) \\ &= f' - f_1\theta^1 - f_2\theta^2 + q^{-2}f_{21}\theta^2\theta^1. \end{aligned} \quad (63)$$

Our considerations about translations can also be applied to the opposite Hopf structure given by

$$\Delta_{\bar{R}} \equiv \tau \circ \Delta_{\bar{L}}, \quad S_{\bar{R}} \equiv S_{\bar{L}}^{-1}, \quad (64)$$

where τ denotes transposition of tensor factors. Right and left coordinates are now defined by

$$\theta_l^\alpha \equiv \theta^\beta \otimes (\bar{\mathcal{L}}_\theta)_\beta^\alpha, \quad \theta_r^\alpha \equiv 1 \otimes \theta^\alpha. \quad (65)$$

Then we have for coproduct and antipode respectively

$$\begin{aligned} \Delta_{\bar{R}}(\theta^\alpha) &= \theta_l^\alpha + \theta_r^\alpha, \quad \alpha = 1, 2, \\ \Delta_{\bar{R}}(\theta^1\theta^2) &= \theta_l^1\theta_l^2 + \theta_l^1\theta_r^2 - q\theta_l^2\theta_r^1 + \theta_r^1\theta_r^2, \end{aligned} \quad (66)$$

and

$$\begin{aligned} S_{\bar{R}}(\theta^\alpha) &= -\theta^\alpha, \quad \alpha = 1, 2, \\ S_{\bar{R}}(\theta^1\theta^2) &= q^2\theta^1\theta^2. \end{aligned} \quad (67)$$

Consequently, we find as corresponding operations on supernumbers

$$\begin{aligned} f(\theta \underline{\oplus}_{\bar{R}} \psi) &= f' + f_1(\theta^1 + \psi^1) + f_2(\theta^2 + \psi^2) \\ &\quad + f_{12}(\theta^1\theta^2 + \theta^1\psi^2 - q\theta^2\psi^1 + \psi^1\psi^2), \\ f(\underline{\ominus}_{\bar{R}}\theta) &= f' - f_1\theta^1 - f_2\theta^2 + q^2f_{12}\theta^1\theta^2. \end{aligned} \quad (68)$$

From what we have done so far we can easily derive the crossing symmetries

$$\Delta_{\bar{R}}, S_{\bar{R}}, \underline{\oplus}_{\bar{R}}, \underline{\ominus}_{\bar{R}} \xleftrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} \Delta_{\bar{L}}, S_{\bar{L}}, \underline{\oplus}_{\bar{L}}, \underline{\ominus}_{\bar{L}}, \quad (69)$$

with the transition symbol having the same meaning as in Eq. (44).

It would have been possible to begin with the other Hopf structure by starting with the coproducts Δ_L, Δ_R and the antipodes S_L, S_R . However, the expressions for the corresponding operations $\underline{\oplus}_L, \underline{\ominus}_L$ and $\underline{\oplus}_R, \underline{\ominus}_R$ are obtained most easily by the transformation rules

$$\begin{aligned}\Delta_L, S_L, \underline{\oplus}_L, \underline{\ominus}_L &\xrightarrow[q \leftrightarrow 1/q]{a \leftrightarrow \alpha'} \Delta_{\bar{L}}, S_{\bar{L}}, \underline{\oplus}_{\bar{L}}, \underline{\ominus}_{\bar{L}}, \\ \Delta_R, S_R, \underline{\oplus}_R, \underline{\ominus}_R &\xrightarrow[q \leftrightarrow 1/q]{\alpha \leftrightarrow \alpha'} \Delta_{\bar{R}}, S_{\bar{R}}, \underline{\oplus}_{\bar{R}}, \underline{\ominus}_{\bar{R}},\end{aligned}\tag{70}$$

as can be proven by a direct calculation.

Our final comment concerns commutation relations between supernumbers and arbitrary elements of other quantum spaces. In Sec. 2 we called formulae for calculating such relations braided products. Recalling the identities in (19), we can conclude that braided products for supernumbers take on the form

$$\begin{aligned}f(\theta^2, \theta^1) \underline{\odot}_L g &= g \otimes f' + f_\alpha ((\mathcal{L}\theta)_\beta^\alpha \triangleright g) \otimes \theta^\beta \\ &\quad + f_{21} ((\mathcal{L}\theta)_\gamma^2 (\mathcal{L}\theta)_\delta^1 \triangleright g) \otimes \theta^\gamma \theta^\delta, \\ g \underline{\odot}_R f(\theta^2, \theta^1) &= f' \otimes g + \theta^\beta \otimes (g \triangleleft (\mathcal{L}\theta)_\beta^\alpha) f_\alpha \\ &\quad + (g \triangleleft (\mathcal{L}\theta)_\gamma^2 (\mathcal{L}\theta)_\delta^1) \otimes \theta^\gamma \theta^\delta,\end{aligned}\tag{71}$$

and likewise for the other braiding with the \mathcal{L} -operator now substituted by its conjugate. Notice that in the last two identities summation over all repeated indices is to be understood. After having inserted the explicit form of the \mathcal{L} -operator and then rearranging, it follows that (for compactness, we have abbreviated monomials of ordering $\theta^2 \theta^1$ by the symbol $\theta^{\underline{K}}$)

$$\begin{aligned}f(\theta^2, \theta^1) \underline{\odot}_{L/\bar{L}} g &= g \otimes f' + \sum_{\underline{K}} \left((O_f)_{L/\bar{L}}^{\underline{K}} \triangleright g \right) \otimes \theta^{\underline{K}}, \\ g \underline{\odot}_{R/\bar{R}} f(\theta^2, \theta^1) &= f' \otimes g + \sum_{\underline{K}} \theta^{\underline{K}} \otimes \left(g \triangleleft (O_f)_{L/\bar{L}}^{\underline{K}} \right),\end{aligned}\tag{72}$$

where

$$(O_f)_L^1 = \tilde{\Lambda}^{-1} \tau^{-1/4} (f_1 - q \lambda f_2 T^+),\tag{73}$$

$$(O_f)_L^2 = \tilde{\Lambda}^{-1} \tau^{1/4} f_2,$$

$$(O_f)_L^{21} = \tilde{\Lambda}^{-2},\tag{74}$$

and

$$(O_f)_L^1 = \tilde{\Lambda} \tau^{1/4} f_1, \quad (75)$$

$$(O_f)_L^2 = \tilde{\Lambda} \tau^{-1/4} (f_2 + q^{-1} \lambda f_1 T^-),$$

$$(O_f)_L^{21} = \tilde{\Lambda}^2. \quad (76)$$

4 Three-Dimensional q-deformed Euclidean space

The three-dimensional antisymmetrized Euclidean space (for its definition see again Appendix A) can be treated in very much the same way as the two-dimensional quantum plane. Thus we restrict ourselves to stating the results, only. In complete analogy to the two-dimensional case we define left and right Grassmann integrals by

$$\int f(\theta^+, \theta^3, \theta^-) d_L^3 \theta \equiv (\partial_\theta)_- (\partial_\theta)_3 (\partial_\theta)_+ \triangleright f(\theta^+, \theta^3, \theta^-) = f_{+3-}, \quad (77)$$

$$\int d_R^3 \theta f(\theta^-, \theta^3, \theta^+) \equiv f(\theta^-, \theta^3, \theta^+) \triangleleft (\hat{\partial}_\theta)_+ (\hat{\partial}_\theta)_3 (\hat{\partial}_\theta)_- = f_{-3+},$$

where the actions of partial derivatives have been calculated in Ref. [37]. Again this definition has the consequence that

$$\begin{aligned} \int \theta^+ \theta^3 \theta^- d_L^3 \theta &= 1, \quad \int 1 d_L^3 \theta = \int \theta^A d_L^3 \theta = 0, \\ \int \theta^+ \theta^3 d_L^3 \theta &= \int \theta^+ \theta^- d_L^3 \theta = \int \theta^3 \theta^- d_L^3 \theta = 0, \end{aligned} \quad (78)$$

(we take the convention that labels which are not specified any further can take on any of their possible values, i.e. in our case $A \in \{+, 3, -\}$) and

$$\begin{aligned} \int d_R^3 \theta \theta^- \theta^3 \theta^+ &= 1, \quad \int d_R^3 \theta 1 = \int d_R^3 \theta \theta^A = 0, \\ \int d_R^3 \theta \theta^3 \theta^+ &= \int d_R^3 \theta \theta^- \theta^+ = \int d_R^3 \theta \theta^- \theta^3 = 0. \end{aligned} \quad (79)$$

As usual, translational invariance is then given by

$$\begin{aligned} \int (\partial_\theta)_A \triangleright f(\theta^+, \theta^3, \theta^-) d_L^3 \theta &= 0, \\ \int d_R^3 \theta f(\theta^-, \theta^3, \theta^+) \triangleleft (\hat{\partial}_\theta)_A &= 0. \end{aligned} \quad (80)$$

The corresponding expressions arising from the conjugated differential calculus are obtained from the above ones most easily by applying the substitutions

$$\begin{aligned}
d_L^3 \theta &\leftrightarrow d_L^3 \theta, & d_R^3 \theta &\leftrightarrow d_R^3 \theta, \\
f(\theta^+, \theta^3, \theta^-) &\leftrightarrow f(\theta^-, \theta^3, \theta^+), \\
\theta^A &\leftrightarrow \theta^{\bar{A}}, & (\partial_\theta)_A &\leftrightarrow (\hat{\partial}_\theta)_{\bar{A}}, \\
\triangleright &\leftrightarrow \bar{\triangleright}, & \triangleleft &\leftrightarrow \bar{\triangleleft},
\end{aligned} \tag{81}$$

where we have introduced indices with bar by setting

$$(\overline{+, 3, -}) = (-, 3, +). \tag{82}$$

For this substitution to become more clear we give as an example

$$\int (\partial_\theta)_+ \triangleright f(\theta^+, \theta^3, \theta^-) d_L^3 \theta \leftrightarrow \int (\hat{\partial}_\theta)_- \bar{\triangleright} f(\theta^-, \theta^3, \theta^+) d_L^3 \theta. \tag{83}$$

Next we come to the superexponentials. From the pairings

$$\begin{aligned}
\langle (\partial_\theta)_-, \theta^- \rangle_{L, \bar{R}} &= \langle (\partial_\theta)_3, \theta^3 \rangle_{L, \bar{R}} = \langle (\partial_\theta)_+, \theta^+ \rangle_{L, \bar{R}} = 1, \\
\langle (\partial_\theta)_- (\partial_\theta)_3, \theta^3 \theta^- \rangle_{L, \bar{R}} &= 1, & \langle (\partial_\theta)_- (\partial_\theta)_+, \theta^+ \theta^- \rangle_{L, \bar{R}} &= 1, \\
\langle (\partial_\theta)_3 (\partial_\theta)_+, \theta^+ \theta^3 \rangle_{L, \bar{R}} &= 1, & \langle (\partial_\theta)_- (\partial_\theta)_3 (\partial_\theta)_+, \theta^+ \theta^3 \theta^- \rangle_{L, \bar{R}} &= 1,
\end{aligned} \tag{84}$$

we can read off for the exponential an expression which is the same as in the undeformed case:

$$\begin{aligned}
&\exp(\theta_{\bar{R}} \mid (\partial_\theta)_L) \\
&= 1 \otimes 1 + \theta^+ \otimes (\partial_\theta)_+ + \theta^3 \otimes (\partial_\theta)_3 \\
&+ \theta^- \otimes (\partial_\theta)_- + \theta^+ \theta^3 \otimes (\partial_\theta)_3 (\partial_\theta)_+ + \theta^+ \theta^- \otimes (\partial_\theta)_- (\partial_\theta)_+ \\
&+ \theta^3 \theta^- \otimes (\partial_\theta)_- (\partial_\theta)_3 + \theta^+ \theta^3 \theta^- \otimes (\partial_\theta)_- (\partial_\theta)_3 (\partial_\theta)_+.
\end{aligned} \tag{85}$$

In complete accordance with the considerations of the previous section we found as crossing symmetries

$$\langle \hat{\underline{\partial}}_\theta, \underline{\theta} \rangle_{\bar{L}, R} \xleftrightarrow{q \leftrightarrow 1/q} \langle \underline{\partial}_\theta, \theta \rangle_{L, \bar{R}}, \tag{86}$$

$$\langle \underline{\theta}, \hat{\underline{\partial}}_\theta \rangle_{\bar{L}, R} \xleftrightarrow{q \leftrightarrow 1/q} \langle \underline{\theta}, \underline{\partial}_\theta \rangle_{L, \bar{R}},$$

$$\langle \hat{\underline{\partial}}_\theta, \underline{\theta} \rangle_{\bar{L}, R} \xleftrightarrow{q \leftrightarrow 1/q} \langle \underline{\theta}, \hat{\underline{\partial}}_\theta \rangle_{\bar{L}, R}, \tag{87}$$

$$\langle \underline{\partial}_\theta, \underline{\theta} \rangle_{L, \bar{R}} \xleftrightarrow{+ \leftrightarrow -} \langle \underline{\theta}, \underline{\partial}_\theta \rangle_{L, \bar{R}},$$

and

$$\exp(\theta_R \mid (\hat{\partial}_\theta)_{\bar{L}}) \xleftrightarrow[q \leftrightarrow 1/q]{+ \leftrightarrow -} \exp(\theta_{\bar{R}} \mid (\partial_\theta)_L), \quad (88)$$

$$\exp((\hat{\partial}_\theta)_R \mid \theta_{\bar{L}}) \xleftrightarrow[q \leftrightarrow 1/q]{+ \leftrightarrow -} \exp((\partial_\theta)_{\bar{R}} \mid \theta_L),$$

$$\exp(\theta_R \mid (\hat{\partial}_\theta)_{\bar{L}}) \xleftrightarrow{+ \leftrightarrow -} \exp((\hat{\partial}_\theta)_R \mid \theta_{\bar{L}}), \quad (89)$$

$$\exp(\theta_{\bar{R}} \mid (\partial_\theta)_L) \xleftrightarrow{+ \leftrightarrow -} \exp((\partial_\theta)_{\bar{R}} \mid \theta_L).$$

The symbol $\xleftrightarrow[q \leftrightarrow 1/q]{+ \leftrightarrow -}$ now denotes a transition via

$$\begin{aligned} \theta^A &\leftrightarrow \theta^{\bar{A}}, & \theta_A &\leftrightarrow \theta_{\bar{A}}, & q &\leftrightarrow q^{-1}, \\ (\partial_\theta)^A &\leftrightarrow (\hat{\partial}_\theta)^{\bar{A}}, & (\partial_\theta)_A &\leftrightarrow (\hat{\partial}_\theta)_{\bar{A}}, \end{aligned} \quad (90)$$

whereas $\xleftrightarrow{+ \leftrightarrow -}$ stands for one of the following two substitutions:

$$\begin{aligned} \text{a) } &\theta^A \leftrightarrow \hat{\partial}^A, & \hat{\partial}_A &\leftrightarrow \theta_A, \\ \text{b) } &\theta^A \leftrightarrow \partial^A, & \partial_A &\leftrightarrow \theta_A. \end{aligned} \quad (91)$$

Now we concentrate our attention on the Hopf structure for Grassmann variables. With its explicit form given in Ref. [37] we can show that on a basis of normal ordered monomials the expressions for the coproduct become

$$\Delta_L(\theta^A) = \theta_l^A + \theta_r^A, \quad (92)$$

$$\Delta_L(\theta^- \theta^3) = \theta_l^- \theta_l^3 + \theta_l^- \theta_r^3 - q^2 \theta_l^3 \theta_r^- + \theta_r^- \theta_r^3, \quad (93)$$

$$\Delta_L(\theta^- \theta^+) = \theta_l^- \theta_l^+ + \theta_l^- \theta_r^+ - q^4 \theta_l^+ \theta_r^- + \theta_r^- \theta_r^+,$$

$$\Delta_L(\theta^3 \theta^+) = \theta_l^3 \theta_l^+ + \theta_l^3 \theta_r^+ - q^2 \theta_l^+ \theta_r^3 + \theta_r^3 \theta_r^+,$$

$$\begin{aligned} \Delta_L(\theta^- \theta^3 \theta^+) &= \theta_l^- \theta_l^3 \theta_l^+ + \theta_l^- \theta_l^3 \theta_r^+ + q^6 \theta_l^3 \theta_l^+ \theta_r^- - q^2 \theta_l^- \theta_l^+ \theta_r^3 \\ &\quad + \theta_l^- \theta_r^3 \theta_r^+ - q^2 \theta_l^3 \theta_r^- \theta_r^+ + q^6 \theta_l^+ \theta_r^- \theta_r^3 + \theta_r^- \theta_r^3 \theta_r^+. \end{aligned} \quad (94)$$

Notice that right and left coordinates are defined in complete analogy to the two-dimensional case. The above results are consistent with the antipodes

$$\begin{aligned} S_L(\theta^A) &= -\theta^A, \\ S_L(\theta^- \theta^3) &= q^4 \theta^- \theta^3, & S_L(\theta^- \theta^+) &= q^4 \theta^- \theta^+, \end{aligned} \quad (95)$$

$$S_L(\theta^3\theta^+) = q^4\theta^3\theta^+, \quad S_L(\theta^-\theta^3\theta^+) = q^8\theta^-\theta^3\theta^+.$$

Making use of the crossing symmetries

$$\Delta_L, S_L \xrightleftharpoons[q \leftrightarrow 1/q]{+\leftrightarrow -} \Delta_{\bar{L}}, S_{\bar{L}}, \quad (96)$$

$$\Delta_R, S_R \xrightleftharpoons[q \leftrightarrow 1/q]{+\leftrightarrow -} \Delta_{\bar{R}}, S_{\bar{R}},$$

$$\Delta_L, S_L \xrightleftharpoons[q \leftrightarrow 1/q]{+\leftrightarrow -} \Delta_R, S_R, \quad (97)$$

$$\Delta_{\bar{L}}, S_{\bar{L}} \xrightleftharpoons[q \leftrightarrow 1/q]{+\leftrightarrow -} \Delta_{\bar{R}}, S_{\bar{R}},$$

we are able to find the corresponding expressions for the other types of Hopf structures. Now, the explicit form of the q -deformed addition law for supernumbers should be rather apparent from what we have done so far. Thus it is left to the reader to write down the explicit form of the operations $\underline{\oplus}$ and $\underline{\ominus}$.

We conclude this section by presenting explicit formulae for braided products concerning supernumbers represented in the form of Eq. (4), where $\theta^{\underline{K}}$ shall now denote monomials of ordering $\theta^+\theta^3\theta^-$. Explicitly, we have

$$f(\theta^+, \theta^3, \theta^-) \underline{\ominus}_{L/\bar{L}} g = g \otimes f' + \sum_{\underline{K}} \left((O_f)_{L/\bar{L}}^{\underline{K}} \triangleright g \right) \otimes \theta^{\underline{K}}, \quad (98)$$

$$g \underline{\ominus}_{R/\bar{R}} f(\theta^+, \theta^3, \theta^-) = f' \otimes g + \sum_{\underline{K}} \theta^{\underline{K}} \otimes \left(g \triangleleft (O_f)_{L/\bar{L}}^{\underline{K}} \right),$$

where we introduced abbreviations for the following combinations of $U_q(su_2)$ -generators:

$$(O_f)_L^+ = \tilde{\Lambda}^{-1} f_+ \tau^{1/2}, \quad (99)$$

$$(O_f)_L^3 = \tilde{\Lambda}^{-1} (f_3 + q\lambda\lambda_+ f_+ \tau^{1/2} L^+),$$

$$(O_f)_L^- = \tilde{\Lambda}^{-1} (f_- \tau^{1/2} + \lambda\lambda_+ f_3 L^+ + q^2 \lambda^2 \lambda_+ f_+ \tau^{1/2} (L^+)^2),$$

$$(O_f)_L^{+3} = \tilde{\Lambda}^{-2} f_{+3} \tau^{1/2}, \quad (100)$$

$$(O_f)_L^{+-} = \tilde{\Lambda}^{-2} (f_{+-} + q^2 \lambda\lambda_+ f_{+3} \tau^{1/2} L^+),$$

$$(O_f)_L^{3-} = \tilde{\Lambda}^{-2} (f_{3-} \tau^{-1/2} - q^{-1} \lambda\lambda_+ f_{+-} L^+ + q^2 \lambda^2 \lambda_+ f_{+3} \tau^{1/2} (L^+)^2),$$

$$(O_f)_L^{+3-} = \tilde{\Lambda}^{-3} f_{+3-}, \quad (101)$$

and likewise for the second braiding

$$\begin{aligned}(O_f)_{\bar{L}}^+ &= \tilde{\Lambda}(f_+ \tau^{-1/2} + \lambda \lambda_+ f_3 L^- + q^{-2} \lambda^2 \lambda_+ f_- \tau^{1/2} (L^-)^2), \\ (O_f)_{\bar{L}}^3 &= \tilde{\Lambda}(f_3 + q^{-1} \lambda \lambda_+ f_- \tau^{1/2} L^-), \\ (O_f)_{\bar{L}}^- &= \tilde{\Lambda} f_- \tau^{-1/2},\end{aligned}\tag{102}$$

$$\begin{aligned}(O_f)_{\bar{L}}^{+3} &= \tilde{\Lambda}^2(f_{+3} \tau^{-1/2} - q^{-1} \lambda \lambda_+ f_{+-} L^- \\ &\quad + q^{-2} \lambda^2 \lambda_+ f_{3-} \tau^{1/2} (L^-)^2), \\ (O_f)_{\bar{L}}^{+-} &= \tilde{\Lambda}^2(f_{+-} + \lambda \lambda_+ f_{3-} \tau^{1/2} L^-), \\ (O_f)_{\bar{L}}^{3-} &= \tilde{\Lambda}^2 f_{3-} \tau^{1/2},\end{aligned}\tag{103}$$

$$(O_f)_{\bar{L}}^{+3-} = \tilde{\Lambda}^3 f_{+3-}.\tag{104}$$

As in the two-dimensional case our formulae for braided products require to know the actions of symmetry generators on quantum space elements. The explicit form of these actions is already known from Refs. [33] and [37].

5 Four-Dimensional q-deformed Euclidean space

All considerations of the previous sections pertain equally to the antisymmetrized Euclidean space with four dimensions. For its definition and some basic results used in the following we refer the reader to Appendix A and Ref. [37]. To begin, we introduce Grassmann integrals by

$$\begin{aligned}\int f(\theta^4, \theta^3, \theta^2, \theta^1) d_L^4 \theta \\ \equiv (\partial_\theta)_1 (\partial_\theta)_2 (\partial_\theta)_3 (\partial_\theta)_4 \triangleright f(\theta^4, \theta^3, \theta^2, \theta^1) = f_{1234},\end{aligned}\tag{105}$$

$$\begin{aligned}\int d_R^4 \theta f(\theta^1, \theta^2, \theta^3, \theta^4) \\ \equiv f(\theta^1, \theta^2, \theta^3, \theta^4) \triangleleft (\hat{\partial}_\theta)_4 (\hat{\partial}_\theta)_3 (\hat{\partial}_\theta)_2 (\hat{\partial}_\theta)_1 = f_{4321}.\end{aligned}\tag{106}$$

Using the explicit form for the action of partial derivatives (see Ref. [37]), one immediately arrives at

$$\begin{aligned}\int \theta^4 \theta^3 \theta^2 \theta^1 d_L^4 \theta = 1, \quad \int \theta^i d_L^4 \theta = 0, \quad i = 1, \dots, 4, \\ \int \theta^i \theta^j d_L^4 \theta = 0, \quad (i, j) \in \{(4, 3), (4, 2), (4, 1), (3, 2), (3, 1), (2, 1)\},\end{aligned}\tag{107}$$

$$\int \theta^k \theta^l \theta^m d_L^4 \theta = 0, \quad (k, l, m) \in \{(4, 3, 2), (4, 3, 1), (4, 2, 1), (3, 2, 1)\},$$

and

$$\int d_R^4 \theta \theta^1 \theta^2 \theta^3 \theta^4 = 1, \quad \int d_R^4 \theta \theta^i = 0, \quad (108)$$

$$\int d_R^4 \theta \theta^i \theta^j = 0, \quad (i, j) \in \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\},$$

$$\int d_R^4 \theta \theta^k \theta^l \theta^m = 0, \quad (k, l, m) \in \{(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)\}.$$

In the same way one readily proves translational invariance, i.e.

$$\begin{aligned} \int (\partial_\theta)_i \triangleright f(\theta^4, \theta^3, \theta^2, \theta^1) d_L^4 \theta &= 0, \\ \int d_R^4 \theta f(\theta^1, \theta^2, \theta^3, \theta^4) \triangleleft (\hat{\partial}_\theta)_i &= 0. \end{aligned} \quad (109)$$

Applying the substitutions

$$\begin{aligned} d_L^4 \theta &\leftrightarrow d_L^4 \theta, \quad d_R^4 \theta \leftrightarrow d_R^4 \theta, \\ f(\theta^1, \theta^2, \theta^3, \theta^4) &\leftrightarrow f(\theta^4, \theta^3, \theta^2, \theta^1), \\ \theta^i &\leftrightarrow \theta^{i'}, \quad (\partial_\theta)_i \leftrightarrow (\hat{\partial}_\theta)_{i'}, \quad i' \equiv i - 5, \\ \triangleright &\leftrightarrow \bar{\triangleright}, \quad \triangleleft \leftrightarrow \bar{\triangleleft}, \end{aligned} \quad (110)$$

to all of the above expressions yields the corresponding identities for the conjugated differential calculus.

Next, we turn to superexponentials. From the dual pairings

$$\langle (\partial_\theta)_i, \theta^i \rangle_{L, \bar{R}} = 1, \quad i = 1 \dots, 4, \quad (111)$$

$$\begin{aligned} \langle (\partial_\theta)_1 (\partial_\theta)_2, \theta^2 \theta^1 \rangle_{L, \bar{R}} &= \langle (\partial_\theta)_1 (\partial_\theta)_3, \theta^3 \theta^1 \rangle_{L, \bar{R}} \\ &= \langle (\partial_\theta)_1 (\partial_\theta)_4, \theta^4 \theta^1 \rangle_{L, \bar{R}} = \langle (\partial_\theta)_2 (\partial_\theta)_3, \theta^3 \theta^2 \rangle_{L, \bar{R}} \\ &= \langle (\partial_\theta)_2 (\partial_\theta)_4, \theta^4 \theta^2 \rangle_{L, \bar{R}} = \langle (\partial_\theta)_3 (\partial_\theta)_4, \theta^4 \theta^3 \rangle_{L, \bar{R}} = 1, \end{aligned} \quad (112)$$

$$\begin{aligned} \langle (\partial_\theta)_1 (\partial_\theta)_2 (\partial_\theta)_3, \theta^3 \theta^2 \theta^1 \rangle_{L, \bar{R}} &= \langle (\partial_\theta)_1 (\partial_\theta)_2 (\partial_\theta)_4, \theta^4 \theta^2 \theta^1 \rangle_{L, \bar{R}} \\ &= \langle (\partial_\theta)_1 (\partial_\theta)_3 (\partial_\theta)_4, \theta^4 \theta^3 \theta^1 \rangle_{L, \bar{R}} = \langle (\partial_\theta)_2 (\partial_\theta)_3 (\partial_\theta)_4, \theta^4 \theta^3 \theta^2 \rangle_{L, \bar{R}} = 1, \end{aligned} \quad (113)$$

$$\langle (\partial_\theta)_1 (\partial_\theta)_2 (\partial_\theta)_3 (\partial_\theta)_4, \theta^4 \theta^3 \theta^2 \theta^1 \rangle_{L, \bar{R}} = 1, \quad (114)$$

we can deduce for the exponential

$$\begin{aligned}
& \exp(\theta_{\bar{R}} \mid (\partial_\theta)_L) \\
&= 1 \otimes 1 + \theta^1 \otimes (\partial_\theta)_1 + \theta^2 \otimes (\partial_\theta)_2 \\
&+ \theta^3 \otimes (\partial_\theta)_3 + \theta^4 \otimes (\partial_\theta)_4 + \theta^4 \theta^3 \otimes (\partial_\theta)_3 (\partial_\theta)_4 \\
&+ \theta^4 \theta^2 \otimes (\partial_\theta)_2 (\partial_\theta)_4 + \theta^4 \theta^1 \otimes (\partial_\theta)_1 (\partial_\theta)_4 \\
&+ \theta^3 \theta^1 \otimes (\partial_\theta)_1 (\partial_\theta)_3 + \theta^2 \theta^1 \otimes (\partial_\theta)_1 (\partial_\theta)_2 \\
&+ \theta^4 \theta^3 \theta^2 \otimes (\partial_\theta)_2 (\partial_\theta)_3 (\partial_\theta)_4 + \theta^4 \theta^3 \theta^1 \otimes (\partial_\theta)_1 (\partial_\theta)_3 (\partial_\theta)_4 \\
&+ \theta^4 \theta^2 \theta^1 \otimes (\partial_\theta)_1 (\partial_\theta)_2 (\partial_\theta)_4 + \theta^3 \theta^2 \theta^1 \otimes (\partial_\theta)_1 (\partial_\theta)_2 (\partial_\theta)_3 \\
&+ \theta^4 \theta^3 \theta^2 \theta^1 \otimes (\partial_\theta)_1 (\partial_\theta)_2 (\partial_\theta)_3 (\partial_\theta)_4.
\end{aligned} \tag{115}$$

The other types of pairings and exponentials correspond to the above expressions through

$$\langle \hat{\partial}_\theta, \underline{\theta} \rangle_{\bar{L}, R} \xleftrightarrow[q \leftrightarrow 1/q]{i \leftrightarrow i'} \langle \underline{\partial}_\theta, \theta \rangle_{L, \bar{R}}, \tag{116}$$

$$\langle \underline{\theta}, \hat{\partial}_\theta \rangle_{\bar{L}, R} \xleftrightarrow[q \leftrightarrow 1/q]{i \leftrightarrow i'} \langle \underline{\theta}, \underline{\partial}_\theta \rangle_{L, \bar{R}},$$

$$\langle \hat{\partial}_\theta, \underline{\theta} \rangle_{\bar{L}, R} \xleftrightarrow{i \leftrightarrow i'} \langle \underline{\theta}, \hat{\partial}_\theta \rangle_{\bar{L}, R}, \tag{117}$$

$$\langle \underline{\partial}_\theta, \underline{\theta} \rangle_{L, \bar{R}} \xleftrightarrow{i \leftrightarrow i'} \langle \underline{\theta}, \underline{\partial}_\theta \rangle_{L, \bar{R}},$$

and

$$\exp(\theta_R \mid (\hat{\partial}_\theta)_{\bar{L}}) \xleftrightarrow[q \leftrightarrow 1/q]{i \leftrightarrow i'} \exp(\theta_{\bar{R}} \mid (\partial_\theta)_L), \tag{118}$$

$$\exp((\hat{\partial}_\theta)_R \mid \theta_{\bar{L}}) \xleftrightarrow[q \leftrightarrow 1/q]{i \leftrightarrow i'} \exp((\partial_\theta)_{\bar{R}} \mid \theta_L),$$

$$\exp(\theta_R \mid (\hat{\partial}_\theta)_{\bar{L}}) \xleftrightarrow{i \leftrightarrow i'} \exp((\hat{\partial}_\theta)_R \mid \theta_{\bar{L}}), \tag{119}$$

$$\exp(\theta_{\bar{R}} \mid (\partial_\theta)_L) \xleftrightarrow{i \leftrightarrow i'} \exp((\partial_\theta)_{\bar{R}} \mid \theta_L),$$

where $\xleftrightarrow[q \leftrightarrow 1/q]{+ \leftrightarrow -}$ symbolizes the transition

$$\begin{aligned}
\theta^i &\leftrightarrow \theta^{i'}, & \theta_i &\leftrightarrow \theta_{i'}, & q &\leftrightarrow q^{-1}, \\
(\partial_\theta)^i &\leftrightarrow (\hat{\partial}_\theta)^{i'}, & (\partial_\theta)_i &\leftrightarrow (\hat{\partial}_\theta)_{i'},
\end{aligned} \tag{120}$$

and $\overset{+}{\longleftrightarrow}\overset{-}{\longrightarrow}$ stands for one of the following two substitutions:

$$\begin{aligned} \text{a) } & \theta^i \leftrightarrow \hat{\partial}^i, \quad \hat{\partial}_i \leftrightarrow \theta_i, \\ \text{b) } & \theta^i \leftrightarrow \partial^i, \quad \partial_i \leftrightarrow \theta_i. \end{aligned} \quad (121)$$

As we already know, Grassmann translations are determined by the explicit form of the coproduct, for which we found in terms of right and left coordinates

$$\Delta_L(\theta^i) = \theta_l^i + \theta_r^i, \quad i = 1, \dots, 4, \quad (122)$$

$$\begin{aligned} \Delta_L(\theta^j \theta^k) &= \theta_l^j \theta_l^k + \theta_l^j \theta_r^k - q \theta_l^k \theta_r^j + \theta_r^j \theta_r^k, \\ \Delta_L(\theta^1 \theta^4) &= \theta_l^1 \theta_l^4 + \theta_l^1 \theta_r^4 - q^2 \theta_l^4 \theta_r^1 + \theta_r^1 \theta_r^4, \\ \Delta_L(\theta^2 \theta^3) &= \theta_l^2 \theta_l^3 + \theta_l^2 \theta_r^3 - q^2 \theta_l^3 \theta_r^2 + \theta_r^2 \theta_r^3 - q^2 \lambda \theta_l^4 \theta_r^1, \\ \Delta_L(\theta^1 \theta^2 \theta^3) &= \theta_l^1 \theta_l^2 \theta_l^3 + \theta_l^1 \theta_l^2 \theta_r^3 - q^2 \theta_l^1 \theta_l^3 \theta_r^2 \\ &\quad + q^2 \theta_l^2 \theta_l^3 \theta_r^1 + \theta_l^1 \theta_r^2 \theta_r^3 - q \theta_l^2 \theta_r^1 \theta_r^3 \\ &\quad + q^3 \theta_l^3 \theta_r^1 \theta_r^2 + \theta_r^1 \theta_r^2 \theta_r^3 - q^2 \lambda \theta_l^1 \theta_l^4 \theta_r^1, \\ \Delta_L(\theta^1 \theta^a \theta^4) &= \theta_l^1 \theta_l^a \theta_l^4 + \theta_l^1 \theta_l^a \theta_r^4 - q \theta_l^1 \theta_l^4 \theta_r^a \\ &\quad + q^3 \theta_l^a \theta_l^4 \theta_r^1 - q \theta_l^a \theta_r^1 \theta_r^4 + q^3 \theta_l^4 \theta_r^1 \theta_r^a \\ &\quad + \theta_l^1 \theta_r^a \theta_r^4 + \theta_r^1 \theta_r^a \theta_r^4, \\ \Delta_L(\theta^2 \theta^3 \theta^4) &= \theta_l^2 \theta_l^3 \theta_l^4 + \theta_l^2 \theta_l^3 \theta_r^4 - q^2 \theta_l^3 \theta_r^2 \theta_r^4 \\ &\quad + q^2 \theta_l^4 \theta_r^2 \theta_r^3 + \theta_l^2 \theta_r^3 \theta_r^4 - q \theta_l^2 \theta_l^4 \theta_r^3 \\ &\quad + q^3 \theta_l^3 \theta_l^4 \theta_r^2 + \theta_r^2 \theta_r^3 \theta_r^4 - q^2 \lambda \theta_l^1 \theta_r^1 \theta_r^4, \\ \Delta_L(\theta^1 \theta^2 \theta^3 \theta^4) &= \theta_l^1 \theta_l^2 \theta_l^3 \theta_l^4 + \theta_l^1 \theta_l^2 \theta_l^3 \theta_r^4 - q \theta_l^1 \theta_l^2 \theta_l^4 \theta_r^3 \\ &\quad + q^3 \theta_l^1 \theta_l^3 \theta_l^4 \theta_r^2 - q^4 \theta_l^2 \theta_l^3 \theta_l^4 \theta_r^1 + \theta_l^1 \theta_l^2 \theta_r^3 \theta_r^4 \\ &\quad - q^2 \theta_l^1 \theta_l^3 \theta_r^2 \theta_r^4 + q^2 \theta_l^1 \theta_l^4 \theta_r^2 \theta_r^3 + q^2 \theta_l^2 \theta_l^3 \theta_r^1 \theta_r^4 \\ &\quad - q^4 \theta_l^2 \theta_l^4 \theta_r^1 \theta_r^3 + q^6 \theta_l^3 \theta_l^4 \theta_r^1 \theta_r^2 + \theta_l^1 \theta_r^2 \theta_r^3 \theta_r^4 \\ &\quad - q \theta_l^2 \theta_r^1 \theta_r^3 \theta_r^4 + q^3 \theta_l^3 \theta_r^1 \theta_r^2 \theta_r^4 - q^4 \theta_l^4 \theta_r^1 \theta_r^2 \theta_r^3 \\ &\quad + \theta_r^1 \theta_r^2 \theta_r^3 \theta_r^4 - q^2 \lambda \theta_l^1 \theta_l^4 \theta_r^1 \theta_r^4, \end{aligned} \quad (123)$$

$$\begin{aligned} \Delta_L(\theta^1 \theta^2 \theta^3 \theta^4) &= \theta_l^1 \theta_l^2 \theta_l^3 \theta_l^4 + \theta_l^1 \theta_l^2 \theta_l^3 \theta_r^4 - q \theta_l^1 \theta_l^2 \theta_l^4 \theta_r^3 \\ &\quad + q^3 \theta_l^1 \theta_l^3 \theta_l^4 \theta_r^2 - q^4 \theta_l^2 \theta_l^3 \theta_l^4 \theta_r^1 + \theta_l^1 \theta_l^2 \theta_r^3 \theta_r^4 \\ &\quad - q^2 \theta_l^1 \theta_l^3 \theta_r^2 \theta_r^4 + q^2 \theta_l^1 \theta_l^4 \theta_r^2 \theta_r^3 + q^2 \theta_l^2 \theta_l^3 \theta_r^1 \theta_r^4 \\ &\quad - q^4 \theta_l^2 \theta_l^4 \theta_r^1 \theta_r^3 + q^6 \theta_l^3 \theta_l^4 \theta_r^1 \theta_r^2 + \theta_l^1 \theta_r^2 \theta_r^3 \theta_r^4 \\ &\quad - q \theta_l^2 \theta_r^1 \theta_r^3 \theta_r^4 + q^3 \theta_l^3 \theta_r^1 \theta_r^2 \theta_r^4 - q^4 \theta_l^4 \theta_r^1 \theta_r^2 \theta_r^3 \\ &\quad + \theta_r^1 \theta_r^2 \theta_r^3 \theta_r^4 - q^2 \lambda \theta_l^1 \theta_l^4 \theta_r^1 \theta_r^4, \end{aligned} \quad (124)$$

where

$$a = 2, 3, \quad (j, k) \in \{(1, 2), (1, 3), (2, 4), (3, 4)\}.$$

For the sake of completeness we wish to write down the expressions the accompanying antipode gives on the same basis of normal ordered monomials:

$$S_L(\theta^i) = -\theta^i, \quad i = 1, \dots, 4, \quad (125)$$

$$\begin{aligned}
S_L(\theta^j \theta^k) &= q^2 \theta^j \theta^k, \\
S_L(\theta^l \theta^m \theta^n) &= -q^6 \theta^l \theta^m \theta^n, \\
S_L(\theta^1 \theta^2 \theta^3 \theta^4) &= q^{12} \theta^1 \theta^2 \theta^3 \theta^4,
\end{aligned}$$

with

$$\begin{aligned}
(i, j) &\in \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}, \\
(l, m, n) &\in \{(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)\}.
\end{aligned} \tag{126}$$

The formulae for the other types of Hopf structures follow from

$$\Delta_L, S_L \xrightleftharpoons[q \leftrightarrow 1/q]{i \leftrightarrow i'} \Delta_{\bar{L}}, S_{\bar{L}}, \tag{127}$$

$$\Delta_R, S_R \xrightleftharpoons[q \leftrightarrow 1/q]{i \leftrightarrow i'} \Delta_{\bar{R}}, S_{\bar{R}},$$

$$\Delta_L, S_L \xrightleftharpoons[q \leftrightarrow 1/q]{i \leftrightarrow i'} \Delta_R, S_R, \tag{128}$$

$$\Delta_{\bar{L}}, S_{\bar{L}} \xrightleftharpoons[q \leftrightarrow 1/q]{i \leftrightarrow i'} \Delta_{\bar{R}}, S_{\bar{R}}.$$

Last but not least we list expressions for braided products with super-numbers. In general, we have

$$\begin{aligned}
f(\theta^1, \theta^2, \theta^3, \theta^4) \underline{\odot}_{L/\bar{L}} g &= g \otimes f' + \sum_{\underline{K}} \left((O_f)_{L/\bar{L}}^{\underline{K}} \triangleright g \right) \otimes \theta^{\underline{K}}, \\
g \underline{\odot}_{R/\bar{R}} f(\theta^1, \theta^2, \theta^3, \theta^4) &= f' \otimes g + \sum_{\underline{K}} \theta^{\underline{K}} \otimes \left(g \triangleleft (O_f)_{L/\bar{L}}^{\underline{K}} \right),
\end{aligned} \tag{129}$$

where the sum includes all monomials of ordering $\theta^1 \theta^2 \theta^3 \theta^4$, and the operators we introduced in the above formulae are specified by the following combinations of $U_q(so_4)$ -generators (for their action on quantum space elements we refer again to Refs. [33] and [37]):

$$\begin{aligned}
(O_f)_L^1 &= \tilde{\Lambda} K_1^{1/2} K_2^{1/2} (f_1 + q\lambda f_2 L_1^+ + q\lambda f_3 L_2^+ \\
&\quad - q^2 \lambda^2 f_4 L_1^+ L_2^+),
\end{aligned} \tag{130}$$

$$(O_f)_L^2 = \tilde{\Lambda} K_1^{-1/2} K_2^{1/2} (f_2 - q\lambda f_4 L_2^+),$$

$$(O_f)_L^3 = \tilde{\Lambda} K_1^{1/2} K_2^{-1/2} (f_3 - q\lambda f_4 L_1^+),$$

$$(O_f)_L^4 = \tilde{\Lambda} K_1^{-1/2} K_2^{-1/2} f_4,$$

$$(O_f)_L^{12} = \tilde{\Lambda}^2 K_2 (f_{12} - q\lambda f_{14} L_2^+ - q^2 \lambda f_{23} L_2^+ \tag{131}$$

$$\begin{aligned}
& - q\lambda^2 f_{34}(L_2^+)^2), \\
(O_f)_L^{13} &= \tilde{\Lambda}^2 K_1 (f_{13} - q\lambda f_{14} L_1^+ + \lambda f_{23} L_1^+ \\
& - q\lambda^2 f_{24}(L_1^+)^2), \\
(O_f)_L^{14} &= \tilde{\Lambda}^2 (f_{14} - q^2 \lambda f_{24} L_1^+ + \lambda f_{34} L_2^+), \\
(O_f)_L^{23} &= \tilde{\Lambda}^2 (f_{23} - q\lambda f_{24} L_1^+ + q\lambda f_{34} L_2^+), \\
(O_f)_L^{24} &= f_{24} \tilde{\Lambda}^2 K_1^{-1}, \\
(O_f)_L^{34} &= f_{34} \tilde{\Lambda}^2 K_2^{-1}, \\
(O_f)_L^{123} &= \tilde{\Lambda}^3 K_1^{1/2} K_2^{1/2} (f_{123} - q\lambda f_{124} L_1^+ - q\lambda f_{134} L_2^+ \\
& + q^2 \lambda^2 f_{234} L_1^+ L_2^+), \\
(O_f)_L^{124} &= \tilde{\Lambda}^3 K_1^{-1/2} K_2^{1/2} (f_{124} - q\lambda f_{234} L_2^+), \\
(O_f)_L^{134} &= \tilde{\Lambda}^3 K_1^{1/2} K_2^{-1/2} (f_{134} + q\lambda f_{234} L_1^+), \\
(O_f)_L^{234} &= f_{234} \tilde{\Lambda}^3 K_1^{-1/2} K_2^{-1/2}, \\
(O_f)_L^{1234} &= f_{1234} \tilde{\Lambda}^4,
\end{aligned} \tag{132}$$

$$\tag{133}$$

and likewise for the other braiding

$$\begin{aligned}
(O_f)_L^1 &= \tilde{\Lambda}^{-1} f_1 K_1^{-1/2} K_2^{-1/2}, \\
(O_f)_L^2 &= \tilde{\Lambda}^{-1} K_1^{1/2} K_2^{-1/2} (f_2 - q^{-1} \lambda f_1 L_1^-), \\
(O_f)_L^3 &= \tilde{\Lambda}^{-1} K_1^{-1/2} K_2^{1/2} (f_3 - q^{-1} \lambda f_1 L_2^-), \\
(O_f)_L^4 &= \tilde{\Lambda}^{-1} K_1^{1/2} K_2^{1/2} (f_4 + q^{-1} \lambda f_3 L_1^- - q^{-1} \lambda f_2 L_2^- \\
& - q^{-2} \lambda^2 f_1 L_1^- L_2^-),
\end{aligned} \tag{134}$$

$$\begin{aligned}
(O_f)_L^{12} &= f_{12} \tilde{\Lambda}^{-2} K_2^{-1}, \\
(O_f)_L^{13} &= f_{13} \tilde{\Lambda}^{-2} K_1^{-1}, \\
(O_f)_L^{14} &= \tilde{\Lambda}^{-2} (f_{14} + q^{-1} \lambda f_{13} L_1^- + q^{-3} \lambda f_{12} L_2^-), \\
(O_f)_L^{23} &= \tilde{\Lambda}^{-2} (f_{23} - q^{-2} \lambda f_{13} L_1^- + q^{-2} \lambda f_{12} L_2^-), \\
(O_f)_L^{24} &= \tilde{\Lambda}^{-2} K_1 (f_{24} - \lambda f_{14} L_1^- - q^{-1} \lambda f_{23} L_1^- \\
& - q^{-1} \lambda^2 f_{13} (L_1^-)^2), \\
(O_f)_L^{34} &= \tilde{\Lambda}^{-2} K_2 (f_{34} - \lambda f_{14} L_2^- - q\lambda f_{23} L_2^- \\
& - q^{-1} \lambda^2 f_{12} (L_2^-)^2),
\end{aligned} \tag{135}$$

$$(O_f)_{\bar{L}}^{123} = f_{123} \tilde{\Lambda}^{-3} K_1^{-1/2} K_2^{-1/2}, \quad (136)$$

$$(O_f)_{\bar{L}}^{124} = \tilde{\Lambda}^{-3} K_1^{1/2} K_2^{-1/2} (f_{124} + q^{-1} \lambda f_{123} L_1^-),$$

$$(O_f)_{\bar{L}}^{134} = \tilde{\Lambda}^{-3} K_1^{-1/2} K_2^{1/2} (f_{134} - q^{-1} \lambda f_{123} L_2^-),$$

$$(O_f)_{\bar{L}}^{234} = \tilde{\Lambda}^{-3} K_1^{1/2} K_2^{1/2} (f_{234} - q^{-1} \lambda f_{134} L_1^- + q^{-1} \lambda f_{124} L_2^- \\ + q^{-2} \lambda^2 f_{123} L_1^- L_2^-),$$

$$(O_f)_{\bar{L}}^{1234} = f_{1234} \tilde{\Lambda}^{-4}. \quad (137)$$

6 q-Deformed Minkowski space

In this section we would like to focus on antisymmetrised q-Minkowski space (its definition is given in Appendix A). If such a space is fused together with its symmetrized counterpart, it gives a q-deformed superspace useful for physical applications. Again, we start with the introduction of Grassmann integrals:

$$\int f(\theta^-, \theta^{3/0}, \theta^3, \theta^+) d_L^4 \theta \quad (138)$$

$$\equiv -q^{-2} \partial_\theta^- \partial_\theta^0 \partial_\theta^{3/0} \partial_\theta^+ \triangleright f(\theta^-, \theta^{3/0}, \theta^3, \theta^+) = f_{-,3/0,3,+},$$

$$\int d_R^4 \theta f(\theta^-, \theta^{3/0}, \theta^3, \theta^+) \quad (139)$$

$$\equiv -q^2 f(\theta^+, \theta^3, \theta^{3/0}, \theta^-) \triangleleft \hat{\partial}_\theta^+ \hat{\partial}_\theta^{3/0} \hat{\partial}_\theta^0 \hat{\partial}_\theta^- = f_{+,3/0,-}.$$

A direct calculation using the explicit form for the action of partial derivatives on supernumbers [37], shows for left superintegrals that

$$\int \theta^- \theta^{3/0} \theta^3 \theta^+ d_L^4 \theta = 1, \quad (140)$$

$$\int \theta^\mu d_L^4 \theta = 0, \quad \mu \in \{+, 3/0, 3, -\},$$

$$\int \theta^\nu \theta^\rho d_L^4 \theta = 0, \quad \int \theta^\alpha \theta^\beta \theta^\gamma d_L^4 \theta = 0,$$

with

$$(\nu, \rho) \in \{(-, +), (-, 3/0), (-, 3), (3, +), (3, 3/0), (3/0, +)\}, \quad (141)$$

$$(\alpha, \beta, \gamma) \in \{(-, 3/0, 3), (-, 3/0, +), (-, 3, +), (3/0, 3, +)\}.$$

and likewise for right superintegrals,

$$\begin{aligned} \int d_R^4 \theta \theta^+ \theta^3 \theta^{3/0} \theta^- &= 1, \\ \int d_R^4 \theta \theta^\mu &= 0, \quad \mu \in \{+, 3, 3/0, -\}, \\ \int d_R^4 \theta \theta^\nu \theta^\rho &= 0, \quad \int d_R^4 \theta \theta^\alpha \theta^\beta \theta^\gamma = 0, \end{aligned} \quad (142)$$

where

$$\begin{aligned} (\nu, \rho) &\in \{(+, -), (3/0, -), (3, -), (+, 3), (3, 3/0), (+, 3/0)\}, \\ (\alpha, \beta, \gamma) &\in \{(3, 3/0, -), (+, 3/0, -), (+, 3, -), (+, 3, 3/0)\}. \end{aligned} \quad (143)$$

In the same way we can prove translational invariance given by

$$\begin{aligned} \int \partial_\theta^\mu \triangleright f(\theta^-, \theta^{3/0}, \theta^3, \theta^+) d_L^4 \theta &= 0, \\ \int d_R^4 \theta f(\theta^+, \theta^3, \theta^{3/0}, \theta^-) \triangleleft \hat{\partial}_\theta^\mu &= 0. \end{aligned} \quad (144)$$

By performing the substitutions

$$\begin{aligned} d_L^4 \theta &\leftrightarrow d_L^4 \theta, \quad d_R^4 \theta \leftrightarrow d_R^4 \theta, \\ f(\theta^-, \theta^3, \theta^{3/0}, \theta^+) &\leftrightarrow f(\theta^+, \theta^3, \theta^{3/0}, \theta^-), \\ f(\theta^+, \theta^{3/0}, \theta^3, \theta^-) &\leftrightarrow f(\theta^-, \theta^{3/0}, \theta^3, \theta^+), \\ \theta^\pm &\leftrightarrow \theta^\mp, \quad (\partial_\theta)^\pm \leftrightarrow (\hat{\partial}_\theta)^\mp, \quad q \leftrightarrow q^{-1}, \\ \triangleright &\leftrightarrow \bar{\triangleright}, \quad \triangleleft \leftrightarrow \bar{\triangleleft}, \end{aligned} \quad (145)$$

we get the corresponding expressions for the conjugated differential calculus.

It is not very difficult to find out that the two bases described through the pairings below are dual to each other:

$$\begin{aligned} \langle \partial_\theta^+, \theta^- \rangle_{L, \bar{R}} &= -q, \quad \langle \partial_\theta^{3/0}, \theta^3 \rangle_{L, \bar{R}} = 1, \\ \langle \partial_\theta^0, \theta^{3/0} \rangle_{L, \bar{R}} &= 1, \quad \langle \partial_\theta^-, \theta^+ \rangle_{L, \bar{R}} = -q^{-1}, \end{aligned} \quad (146)$$

$$\begin{aligned} \langle \partial_\theta^- \partial_\theta^+, \theta^- \theta^+ \rangle_{L, \bar{R}} &= 1, \quad \langle \partial_\theta^{3/0} \partial_\theta^+, \theta^- \theta^3 \rangle_{L, \bar{R}} = -q, \\ \langle \partial_\theta^- \partial_\theta^{3/0}, \theta^3 \theta^+ \rangle_{L, \bar{R}} &= -q^{-1}, \quad \langle \partial_\theta^0 \partial_\theta^{3/0}, \theta^{3/0} \theta^3 \rangle_{L, \bar{R}} = -q^{-2}, \\ \langle \partial_\theta^0 \partial_\theta^+, \theta^- \theta^0 \rangle_{L, \bar{R}} &= 2q^2 \lambda_+^{-1}, \quad \langle \partial_\theta^- \partial_\theta^0, \theta^0 \theta^+ \rangle_{L, \bar{R}} = 2\lambda_+^{-1}, \end{aligned} \quad (147)$$

$$\langle \partial_\theta^0 \partial_\theta^{3/0} \partial_\theta^+, \theta^- \theta^{3/0} \theta^3 \rangle_{L,\bar{R}} = q^3, \quad \langle \partial_\theta^- \partial_\theta^{3/0} \partial_\theta^+, \theta^- \theta^3 \theta^+ \rangle_{L,\bar{R}} = 1, \quad (148)$$

$$\langle \partial_\theta^- \partial_\theta^0 \partial_\theta^{3/0}, \theta^{3/0} \theta^3 \theta^+ \rangle_{L,\bar{R}} = q, \quad \langle \partial_\theta^- \partial_\theta^0 \partial_\theta^+, \theta^- \theta^0 \theta^+ \rangle_{L,\bar{R}} = -2q\lambda_+^{-1},$$

$$\langle \partial_\theta^- \partial_\theta^{3/0} \partial_\theta^3 \partial_\theta^+, \theta^- \theta^0 \theta^{3/0} \theta^+ \rangle_{L,\bar{R}} = -q^2. \quad (149)$$

By virtue of these identities, the exponential is given by

$$\begin{aligned} & \exp(\theta_{\bar{R}} \mid (\partial_\theta)_L) \\ &= 1 \otimes 1 - q^{-1} \theta^- \otimes \partial_\theta^+ + \theta^3 \otimes \partial_\theta^{3/0} + \theta^{3/0} \otimes \partial_\theta^0 \\ & - q \theta^+ \otimes \partial_\theta^- + \theta^- \theta^+ \otimes \partial_\theta^- \partial_\theta^+ - q^{-1} \theta^- \theta^3 \otimes \partial_\theta^{3/0} \partial_\theta^+ \\ & - q \theta^3 \theta^+ \otimes \partial_\theta^- \partial_\theta^{3/0} - q^{-2} \theta^{3/0} \theta^3 \otimes \partial_\theta^0 \partial_\theta^{3/0} \\ & + \frac{1}{2} q^{-2} \lambda_+ \theta^- \theta^0 \otimes \partial_\theta^0 \partial_\theta^+ + \frac{1}{2} \lambda_+ \theta^0 \theta^+ \otimes \partial_\theta^- \partial_\theta^0 \\ & + q^{-3} \theta^- \theta^{3/0} \theta^3 \otimes \partial_\theta^0 \partial_\theta^{3/0} \partial_\theta^+ + \theta^- \theta^3 \theta^+ \otimes \partial_\theta^- \partial_\theta^{3/0} \partial_\theta^+ \\ & + q^{-1} \theta^{3/0} \theta^3 \theta^+ \otimes \partial_\theta^- \partial_\theta^0 \partial_\theta^{3/0} - \frac{1}{2} q^{-1} \lambda_+ \theta^- \theta^0 \theta^+ \otimes \partial_\theta^- \partial_\theta^0 \partial_\theta^+ \\ & - q^2 \theta^- \theta^{3/0} \theta^3 \theta^+ \otimes \partial_\theta^- \partial_\theta^0 \partial_\theta^{3/0} \partial_\theta^+. \end{aligned} \quad (150)$$

Furthermore, we have the crossing symmetries through

$$\langle \hat{\underline{\partial}}_\theta, \underline{\theta} \rangle_{\bar{L},R} \xleftrightarrow{q \leftrightarrow 1/q} \langle \underline{\partial}_\theta, \theta \rangle_{L,\bar{R}}, \quad (151)$$

$$\langle \underline{\theta}, \hat{\underline{\partial}}_\theta \rangle_{\bar{L},R} \xleftrightarrow{q \leftrightarrow 1/q} \langle \underline{\theta}, \underline{\partial}_\theta \rangle_{L,\bar{R}},$$

$$\langle \hat{\underline{\partial}}_\theta, \underline{\theta} \rangle_{\bar{L},R} \xleftrightarrow{+ \leftrightarrow -} \langle \underline{\theta}, \hat{\underline{\partial}}_\theta \rangle_{\bar{L},R}, \quad (152)$$

$$\langle \underline{\partial}_\theta, \underline{\theta} \rangle_{L,\bar{R}} \xleftrightarrow{+ \leftrightarrow -} \langle \underline{\theta}, \underline{\partial}_\theta \rangle_{L,\bar{R}},$$

and

$$\exp(\theta_R \mid (\hat{\partial}_\theta)_{\bar{L}}) \xleftrightarrow{q \leftrightarrow 1/q} \exp(\theta_{\bar{R}} \mid (\partial_\theta)_L), \quad (153)$$

$$\exp((\hat{\partial}_\theta)_R \mid \theta_{\bar{L}}) \xleftrightarrow{q \leftrightarrow 1/q} \exp((\partial_\theta)_{\bar{R}} \mid \theta_L),$$

$$\exp(\theta_R \mid (\hat{\partial}_\theta)_{\bar{L}}) \xleftrightarrow{+ \leftrightarrow -} \exp((\hat{\partial}_\theta)_R \mid \theta_{\bar{L}}), \quad (154)$$

$$\exp(\theta_{\bar{R}} \mid (\partial_\theta)_L) \xleftrightarrow{+ \leftrightarrow -} \exp((\partial_\theta)_{\bar{R}} \mid \theta_L),$$

where the transition symbols have the very same meaning as in Sec. 4.

Next we would like to provide formulae for the coproduct of Grassmann variables. On a basis of normal ordered monomials we have found

$$\Delta_L(\theta^\mu) = \theta_l^\mu + \theta_r^\mu, \quad \mu \in \{+, 3/0, 0, -, \}, \quad (155)$$

$$\Delta_L(\theta^+ \theta^{3/0}) = \theta_l^+ \theta_l^{3/0} + \theta_l^+ \theta_r^{3/0} - q^{-2} \theta_l^{3/0} \theta_r^+ + \theta_r^+ \theta_r^{3/0}, \quad (156)$$

$$\begin{aligned} \Delta_L(\theta^+ \theta^0) &= \theta_l^+ \theta_l^0 + \theta_l^+ \theta_r^0 - \theta_l^0 \theta_r^+ + \theta_r^+ \theta_r^0 \\ &\quad + \lambda \lambda_+^{-1} \theta_l^+ \theta_r^{3/0} - q^{-2} \lambda \lambda_+^{-1} \theta_l^{3/0} \theta_r^+, \end{aligned}$$

$$\begin{aligned} \Delta_L(\theta^+ \theta^-) &= \theta_l^+ \theta_l^- + \theta_l^+ \theta_r^- - q^{-2} \theta_l^- \theta_r^+ + \theta_r^+ \theta_r^- \\ &\quad + q^{-1} \lambda \lambda_+^{-1} \theta_l^{3/0} \theta_r^{3/0}, \end{aligned}$$

$$\begin{aligned} \Delta_L(\theta^{3/0} \theta^0) &= \theta_l^{3/0} \theta_l^0 + \theta_l^{3/0} \theta_r^0 - q^{-2} \theta_l^0 \theta_r^{3/0} + \theta_r^{3/0} \theta_r^0 \\ &\quad + \lambda \lambda_+^{-1} \theta_l^{3/0} \theta_r^{3/0} - q^{-2} \lambda \theta_l^- \theta_r^+, \end{aligned}$$

$$\Delta_L(\theta^{3/0} \theta^-) = \theta_l^{3/0} \theta_l^- + \theta_l^{3/0} \theta_r^- - q^{-2} \theta_l^- \theta_r^{3/0} + \theta_r^{3/0} \theta_r^-,$$

$$\begin{aligned} \Delta_L(\theta^0 \theta^-) &= \theta_l^0 \theta_l^- + \theta_l^0 \theta_r^- - \theta_l^- \theta_r^0 + \theta_r^0 \theta_r^- \\ &\quad + \lambda \lambda_+^{-1} \theta_l^{3/0} \theta_r^- - q^{-2} \lambda \lambda_+^{-1} \theta_l^- \theta_r^{3/0}, \end{aligned}$$

$$\Delta_L(\theta^+ \theta^{3/0} \theta^0) = \theta_l^+ \theta_l^{3/0} \theta_l^0 + \theta_l^+ \theta_l^{3/0} \theta_r^0 - q^{-2} \theta_l^+ \theta_l^0 \theta_r^{3/0} \quad (157)$$

$$\begin{aligned} &+ q^{-2} \theta_l^{3/0} \theta_l^0 \theta_r^+ + \theta_l^+ \theta_r^{3/0} \theta_r^0 - q^{-2} \theta_l^{3/0} \theta_r^+ \theta_r^0 + q^{-2} \theta_l^0 \theta_r^+ \theta_r^{3/0} \\ &+ \theta_r^+ \theta_r^{3/0} \theta_r^0 + q^{-1} \lambda \theta_l^+ \theta_l^{3/0} \theta_r^{3/0} - q^{-2} \lambda \theta_l^+ \theta_l^- \theta_r^+, \end{aligned}$$

$$\begin{aligned} \Delta_L(\theta^+ \theta^{3/0} \theta^-) &= \theta_l^+ \theta_l^{3/0} \theta_l^- + \theta_l^+ \theta_l^{3/0} \theta_r^- - q^{-2} \theta_l^+ \theta_l^- \theta_r^{3/0} + q^{-4} \theta_l^{3/0} \theta_l^- \theta_r^+ \\ &+ \theta_l^+ \theta_r^{3/0} \theta_r^- - q^{-2} \theta_l^{3/0} \theta_r^+ \theta_r^- + q^{-4} \theta_l^- \theta_r^+ \theta_r^{3/0} + \theta_r^+ \theta_r^{3/0} \theta_r^-, \end{aligned}$$

$$\begin{aligned} \Delta_L(\theta^+ \theta^0 \theta^-) &= \theta_l^+ \theta_l^0 \theta_l^- + \theta_l^+ \theta_l^0 \theta_r^- - \theta_l^+ \theta_l^- \theta_r^0 + q^{-2} \theta_l^0 \theta_l^- \theta_r^+ \\ &+ \theta_l^+ \theta_r^0 \theta_r^- - \theta_l^0 \theta_r^+ \theta_r^- + q^{-2} \theta_l^- \theta_r^+ \theta_r^0 + \theta_r^+ \theta_r^0 \theta_r^- \\ &+ \lambda \lambda_+^{-1} \theta_l^+ \theta_l^{3/0} \theta_r^- + q^{-1} \lambda \lambda_+^{-1} \theta_l^{3/0} \theta_l^0 \theta_r^{3/0} \\ &+ q^{-4} \lambda \lambda_+^{-1} \theta_l^{3/0} \theta_l^- \theta_r^+ + q \lambda (q \lambda - 2) \theta_l^+ \theta_l^- \theta_r^{3/0} \\ &+ \lambda \lambda_+^{-1} \theta_l^+ \theta_r^{3/0} \theta_r^- - 2 q^{-2} \lambda \lambda_+^{-1} \theta_l^{3/0} \theta_r^+ \theta_r^- \\ &- q^{-1} \lambda \lambda_+^{-1} \theta_l^{3/0} \theta_r^{3/0} \theta_r^0 + q^{-4} \lambda \lambda_+^{-1} \theta_l^- \theta_r^+ \theta_r^{3/0}, \end{aligned}$$

$$\begin{aligned} \Delta_L(\theta^{3/0} \theta^0 \theta^-) &= \theta_l^{3/0} \theta_l^0 \theta_l^- + q^{-4} \theta_l^0 \theta_l^- \theta_r^{3/0} - \theta_l^{3/0} \theta_l^- \theta_r^0 \\ &- q^{-2} \theta_l^0 \theta_r^{3/0} \theta_r^+ + q^{-2} \theta_l^- \theta_r^{3/0} \theta_r^0 + \theta_r^{3/0} \theta_r^0 \theta_r^- \\ &+ q^{-3} \lambda \theta_l^{3/0} \theta_l^- \theta_r^{3/0} + q^{-2} \lambda \theta_l^- \theta_r^+ \theta_r^-, \end{aligned}$$

$$\begin{aligned}
\Delta_L(\theta^+\theta^{3/0}\theta^0\theta^-) &= \theta_l^+\theta_l^{3/0}\theta_l^0\theta_l^- + \theta_l^+\theta_l^{3/0}\theta_l^0\theta_r^- - \theta_l^+\theta_l^{3/0}\theta_l^-\theta_r^0 \\
&- q^{-4}\theta_l^{3/0}\theta_l^0\theta_l^-\theta_r^+ + q^{-4}\theta_l^+\theta_l^0\theta_l^-\theta_r^{3/0} + \theta_l^+\theta_l^{3/0}\theta_r^0\theta_r^- \\
&- q^{-2}\theta_l^+\theta_l^0\theta_r^{3/0}\theta_r^- + q^{-2}\theta_l^+\theta_l^-\theta_r^{3/0}\theta_r^0 + q^{-2}\theta_l^{3/0}\theta_l^0\theta_r^+\theta_r^- \\
&- q^{-4}\theta_l^{3/0}\theta_l^-\theta_r^+\theta_r^0 + q^{-6}\theta_l^0\theta_l^-\theta_r^+\theta_r^{3/0} + \theta_l^+\theta_r^{3/0}\theta_r^0\theta_r^- \\
&- q^{-2}\theta_l^{3/0}\theta_r^+\theta_r^0\theta_r^- + q^{-2}\theta_l^0\theta_r^+\theta_r^{3/0}\theta_r^- + q^{-4}\theta_l^-\theta_r^+\theta_r^{3/0}\theta_r^0 \\
&+ \theta_r^+\theta_r^{3/0}\theta_r^0\theta_r^- - 3q^4\lambda\lambda_+^{-1}\theta_l^+\theta_l^{3/0}\theta_l^-\theta_r^0 \\
&+ q^{-2}\lambda\lambda_+^{-1}\theta_l^+\theta_l^{3/0}\theta_r^{3/0}\theta_r^- - q^{-2}\lambda\theta_l^+\theta_l^-\theta_r^+\theta_r^- \\
&- q^6\lambda\lambda_+^{-1}\theta_l^{3/0}\theta_l^-\theta_r^+\theta_r^{3/0} + q^{-2}\lambda\lambda_+^{-1}\theta_l^{3/0}\theta_r^+\theta_r^{3/0}\theta_r^-.
\end{aligned} \tag{158}$$

The expressions for the corresponding antipodes of our normal ordered monomials are:

$$\begin{aligned}
S_L(\theta^\mu) &= -\theta^\mu, \quad \mu \in \{+, 3/0, 0, -\}, \\
S_L(\theta^\nu\theta^\rho) &= q^{-2}\theta^\nu\theta^\rho, \\
S_L(\theta^\alpha\theta^\beta\theta^\gamma) &= -q^{-6}\theta^\alpha\theta^\beta\theta^\gamma, \\
S_L(\theta^+\theta^{3/0}\theta^0\theta^-) &= q^{-12}\theta^+\theta^{3/0}\theta^0\theta^-,
\end{aligned} \tag{159}$$

where

$$\begin{aligned}
(\nu, \rho) &\in \{(+, -), (3/0, -), (0, -), (+, 0), (3/0, 0), (+, 3/0)\}, \\
(\alpha, \beta, \gamma) &\in \{(+, 3/0, 0), (+, 3/0, -), (+, 0, -), (3/0, 0, -)\}.
\end{aligned} \tag{160}$$

Notice that monomials with unspecified indices have to refer to the ordering $\theta^+\theta^{3/0}\theta^0\theta^-$. In complete analogy to the previous sections, one can check the crossing symmetries

$$\Delta_L, S_L \xleftrightarrow[q \leftrightarrow 1/q]{+ \leftrightarrow -} \Delta_{\bar{L}}, S_{\bar{L}}, \tag{161}$$

$$\Delta_R, S_R \xleftrightarrow[q \leftrightarrow 1/q]{+ \leftrightarrow -} \Delta_{\bar{R}}, S_{\bar{R}},$$

$$\Delta_L, S_L \xleftrightarrow[q \leftrightarrow 1/q]{+ \leftrightarrow -} \Delta_R, S_R, \tag{162}$$

$$\Delta_{\bar{L}}, S_{\bar{L}} \xleftrightarrow[q \leftrightarrow 1/q]{+ \leftrightarrow -} \Delta_{\bar{R}}, S_{\bar{R}}.$$

Finally, let us come to expressions for braided products concerning supernumbers. Such braided products can be calculated from

$$f(\theta^+, \theta^{3/0}, \theta^0, \theta^-) \underline{\odot}_{L/\bar{L}} g = g \otimes f' + \sum_{\underline{K}} \left((O_f)_{L/\bar{L}}^{\underline{K}} \triangleright g \right) \otimes \theta^{\underline{K}}, \tag{163}$$

$$g \underline{\odot}_{R/\bar{R}} f(\theta^+, \theta^{3/0}, \theta^0, \theta^-) = f' \otimes g + \sum_{\underline{K}} \theta^{\underline{K}} \otimes \left(g \triangleleft (O_f)_{L/\bar{L}}^{\underline{K}} \right).$$

For brevity, we introduced the following combinations of symmetry generators (for their action on quantum spaces see again Refs. [33] and [37]):

$$\begin{aligned}
(O_f)_L^+ &= \tilde{\Lambda}(\tau^3)^{-1/2} [f_+ \sigma^2 - q^{1/2} \lambda \lambda_+^{1/2} f_{3/0} S^1 \\
&\quad - \lambda^2 f_- T^- S^1 + q^{1/2} \lambda \lambda_+^{-1/2} f_0 (T^- \sigma^2 + q S^1)], \\
(O_f)_L^{3/0} &= \tilde{\Lambda} [-q^{3/2} \lambda \lambda_+^{-1/2} f_+ T^2 + f_{3/0} \tau^1 \\
&\quad - \lambda_+^{-1} f_0 (\lambda^2 T^- T^2 + q(\tau^1 - \sigma^2)) \\
&\quad + q^{-1/2} f_- \lambda \lambda_+^{-1/2} (\tau^1 T^- - q^{-1} S^1)], \\
(O_f)_L^0 &= \tilde{\Lambda} (f_0 \sigma^2 - q^{-1/2} \lambda \lambda_+^{1/2} f_- S^1), \\
(O_f)_L^- &= \tilde{\Lambda} (\tau^3)^{1/2} (-q^{5/2} \lambda \lambda_+^{-1/2} f_0 T^2 + f_- \tau^1), \\
(O_f)_L^{+,3/0} &= \tilde{\Lambda}^2 (\tau^3)^{-1/2} [f_{+,3/0} + q^{-1} \lambda_+^{-1} f_{+0} ((\sigma^2)^2 - 1) \\
&\quad + q^{-1/2} \lambda \lambda_+^{-1/2} f_{+-} (T^- - q^{-1} S^1 \sigma^2) \\
&\quad - q^{-1/2} \lambda \lambda_+^{-1/2} f_{3/0,0} (q^{-1} T^- + s^1 \sigma^2) \\
&\quad + q^{-1} \lambda^2 f_{3/0,-} (S^1)^2 \\
&\quad + \lambda^2 \lambda_{+f0-} ((T^-)^2 - q^{-2} (S^1)^2)], \\
(O_f)_L^{+0} &= \tilde{\Lambda}^2 (\tau^3)^{-1/2} [f_{+0} (\sigma^2)^2 - q^{-1/2} \lambda \lambda_+^{1/2} f_{+-} S^1 \sigma^2 \\
&\quad - q^{1/2} \lambda \lambda_+^{1/2} f_{3/0,0} S^1 \sigma^2 + \lambda^2 \lambda_{+f3/0,-} (S^1)^2 \\
&\quad - q^{-1} \lambda^2 f_{0-} (S^1)^2], \\
(O_f)_L^{+-} &= \tilde{\Lambda}^2 [-q^{1/2} \lambda \lambda_+^{-1/2} f_{+0} T^2 \sigma^2 + f_{+-} (1 + \lambda^2 T^2 S^1) \\
&\quad + q \lambda^2 f_{3/0,0} T^2 S^1 - q^{-3/2} \lambda \lambda_+^{1/2} f_{3/0,-} \tau^1 S^1 \\
&\quad + q^{1/2} \lambda \lambda_+^{-1/2} f_{0-} (T^- + q^{-3} \tau^1 S^1)], \\
(O_f)_L^{3/0,0} &= \tilde{\Lambda}^2 [(-q^{3/2} \lambda \lambda_+^{-1/2} f_{+0} T^2 \sigma^2 + q \lambda^2 f_{+-} T^2 S^1) \\
&\quad + f_{3/0,0} (1 + q^2 \lambda^2 T^2 S^1) - q^{-1/2} \lambda \lambda_+^{1/2} f_{3/0,-} \tau^1 S^1 \\
&\quad + q^{-1/2} \lambda \lambda_+^{-1/2} f_{0-} (T^- + q^{-1} \tau^1 S^1)], \\
(O_f)_L^{3/0,-} &= \tilde{\Lambda}^2 (\tau^3)^{1/2} [q^6 \lambda^2 \lambda_+^{-1} f_{+0} (T^2)^2 \\
&\quad - q^{7/2} \lambda \lambda_+^{-1/2} f_{+-} T^2 \tau^1 + q^{9/2} \lambda \lambda_+^{-1/2} f_{3/0,0} T^2 \tau^1 \\
&\quad + f_{3/0,-} (\tau^1)^2 + q^{-1} \lambda_+^{-1} f_{0-} (1 - (\tau^1)^2)],
\end{aligned} \tag{164}$$

$$\begin{aligned}
(O_f)_L^{0-} &= \tilde{\Lambda}^2(\tau^3)^{-1/2} f_{0-}, \\
(O_f)_L^{+,3/0,0} &= \tilde{\Lambda}^3(\tau^3)^{-1/2} [f_{+,3/0,0} \sigma^2 - q^{-1/2} \lambda \lambda_+^{-1/2} f_{+,3/0,-} S^1 \\
&\quad - q^{1/2} \lambda \lambda_+^{-1/2} f_{+0-} (q T^- \sigma^2 + q^{-1} (q^3 - \lambda_+) S^1) \\
&\quad + \lambda^2 f_{3/0,0,-} T^- S^1], \\
(O_f)_L^{+,3/0,-} &= \tilde{\Lambda}^3 [-q^{5/2} \lambda \lambda_+^{-1/2} f_{+,3/0,0} T^2 + f_{+,3/0,-} \tau^1 \\
&\quad + \lambda_+^{-1} f_{+0-} (q^2 \lambda^2 T^- T^2 + q^{-1} (1 - q^3 \lambda) (\sigma^2 - \tau^1)) \\
&\quad + q^{-3/2} \lambda \lambda_+^{-1/2} f_{3/0,0,-} (T^- \tau^1 + (q^3 - \lambda_+) S^1)], \\
(0_f)_L^{+0-} &= \tilde{\Lambda}^3 (f_{+0-} \sigma^2 - q^{-3/2} \lambda \lambda_+^{1/2} f_{3/0,0,-} S^1), \\
(O_f)_L^{3/0,0-} &= \tilde{\Lambda}^3 (\tau^3)^{1/2} (-q^{7/2} \lambda \lambda_+^{-1/2} f_{+0-} T^2 + f_{3/0,0,-} \tau^1), \\
(O_f)_L^{+,3/0,0-} &= f_{+,3/0,0-} \tilde{\Lambda}^4, \tag{165}
\end{aligned}$$

and

$$\begin{aligned}
(O_f)_L^+ &= \tilde{\Lambda}^{-1} (f_+ \sigma^2 - q^{-1/2} \lambda \lambda_+^{-1/2} f_0 S^1), \tag{166} \\
(O_f)_L^{3/0} &= \tilde{\Lambda}^{-1} (\tau^3)^{-1/2} [-q^{1/2} \lambda \lambda_+^{-1/2} f_+ (T^+ \sigma^2 + q \tau^3 T^2) \\
&\quad + f_{3/0} \sigma^2 + f_0 (\lambda^2 T^+ S^1 + q^{-1} (\tau^3 \tau^1 - \sigma^2)) \\
&\quad - q^{1/2} \lambda \lambda_+^{-1/2} f_- S^1], \\
(O_f)_L^0 &= \tilde{\Lambda}^{-1} (\tau^3)^{1/2} (-q^{1/2} \lambda \lambda_+^{1/2} f_+ T^2 + f_0 (\tau^3)^{-1} \tau^1), \\
(O_f)_L^- &= \tilde{\Lambda}^{-1} [q^2 \lambda^2 f_+ T^2 T^+ - q^{3/2} \lambda \lambda_+^{1/2} f_{3/0} T^2 \\
&\quad - q^{1/2} \lambda \lambda_+^{-1/2} f_0 (q T^+ \tau^1 - T^2)) + f_- \tau^1], \\
(O_f)_L^{+,3/0} &= \tilde{\Lambda}^{-2} (\tau^3)^{-1/2} [f_{+,3/0} (\sigma^2)^2 + q^{-1} \lambda_+^{-1} f_{+0} (\tau^3 - (\sigma^2)^2) \\
&\quad - q^{1/2} \lambda \lambda_+^{-1/2} f_{+-} S^1 \sigma^2 + q^{-1/2} \lambda \lambda_+^{-1/2} f_{3/0,0} S^1 \sigma^2 \\
&\quad + q^2 \lambda^2 \lambda_+^{-1} f_{0-} (S^1)^2], \\
(O_f)_L^{+0} &= \tilde{\Lambda}^{-2} (\tau^3)^{1/2} f_{+0}, \\
(O_f)_L^{+-} &= \tilde{\Lambda}^{-2} [-q^{3/2} \lambda \lambda_+^{1/2} f_{+,3/0} T^2 \sigma^2 \\
&\quad + q^{1/2} \lambda \lambda_+^{-1/2} f_{+0} (T^2 \sigma^2 - q T^+) + f_{+-} (1 + q^2 \lambda^2 T^2 S^1) \\
&\quad - q \lambda^2 f_{3/0,0} T^2 S^1 - q^{3/2} \lambda \lambda_+^{-1/2} f_{0-} \tau^1 S^1], \\
(O_f)_L^{3/0,0} &= \tilde{\Lambda}^{-2} [q^{1/2} \lambda \lambda_+^{1/2} f_{+,3/0} T^2 \sigma^2
\end{aligned}$$

$$\begin{aligned}
& - q\lambda^2 f_{+-} T^2 S^1 + f_{3/0,0} (1 + \lambda^2 T^2 S^1) \\
& + \lambda\lambda_+^{-1/2} f_{+0} (q^{1/2} \tau^1 S^1 - q^{-1/2} (T^2 \sigma^2 + q^3 T^+)), \\
(O_f)_{\bar{L}}^{3/0,-} & = \tilde{\Lambda}^{-2} (\tau^3)^{1/2} [q^3 \lambda^2 f_{+,3/0} (T^2)^2 \\
& + q^2 \lambda^2 \lambda_+^{-1} f_{+0} ((\tau^3)^{-1} (T^+)^2 - (T^2)^2) \\
& - q^{1/2} \lambda \lambda_+^{-1/2} f_{+-} ((\tau^3)^{-1} T^+ + q T^2 \tau^1) \\
& + q^{1/2} \lambda \lambda_+^{-1/2} f_{3/0,0} (T^2 \tau^1 - q (\tau^3)^{-1} T^+) \\
& - q^{-1} \lambda_+^{-1} f_{0-} (1 - (\tau^1)^2)], \\
(O_f)_{\bar{L}}^{0,-} & = \tilde{\Lambda}^{-2} (\tau^3)^{1/2} [q^4 \lambda^2 \lambda_+ f_{+,3/0} (T^2)^2 - q^3 \lambda^2 f_{+0} (T^2)^2 \\
& - q^{5/2} \lambda \lambda_+^{1/2} f_{+-} T^2 \tau^1 + q^{3/2} \lambda \lambda_+^{1/2} f_{3/0,0} T^2 \tau^1 + f_{0-} (\tau^1)^2], \\
(O_f)_{\bar{L}}^{+,3/0,0} & = \tilde{\Lambda}^{-3} (f_{+,3/0,0} \sigma^2 + q^{1/2} \lambda \lambda_+^{-1/2} f_{+0-} S^1) \\
(O_f)_{\bar{L}}^{+,3/0,-} & = \tilde{\Lambda}^{-3} (\tau^3)^{1/2} [q^{1/2} \lambda \lambda_+^{1/2} f_{+,3/0,0} (T^2 - q (\tau^3)^{-1} T^+ \sigma^2) \\
& + f_{+,3/0,-} (\tau^3)^{-1} \sigma^2 + q^{3/2} \lambda \lambda_+^{-1/2} f_{3/0,0-} (\tau^3)^{-1} S^1 \\
& - \lambda_+^{-1} f_{+0-} (q^{-1} (\tau^3)^{-1} \sigma^2 - \tau^1 + q^2 \lambda^2 (\tau^3)^{-1} T^+ S^1)], \\
(O_f)_{\bar{L}}^{+0-} & = \tilde{\Lambda}^{-3} (\tau^3)^{1/2} (q^{3/2} \lambda \lambda_+^{1/2} f_{+,3/0,0} T^2 + f_{+0-} \tau^1), \\
(O_f)_{\bar{L}}^{3/0,0-} & = \tilde{\Lambda}^{-3} [-q^4 \lambda^2 f_{+,3/0,0} T^+ T^2 + q^{1/2} \lambda \lambda_+^{1/2} f_{+,3/0,-} T^2 \\
& - q^{1/2} \lambda \lambda_+^{-1/2} f_{+0-} (q^{-1} T^2 + q^2 T^+ \tau^1) + f_{3/0,0,-} \tau^1], \\
(O_f)_{\bar{L}}^{+,3/0,0-} & = \tilde{\Lambda}^{-4} f_{+,3/0,0-}. \tag{167}
\end{aligned}$$

7 Conclusion

Let us end with a few comments on our results. In the last four sections we have provided q-analogs for elements of superanalysis. In doing so we have realized that on q-deformed quantum spaces Grassmann integrals, Grassmann exponentials and Grassmann translations can be constructed in complete analogy to the classical case. That this analogy is a really far reaching one can also be seen from the fact that translational invariance of our integrals implies rules for integration by parts:

$$\begin{aligned}
\int f(\underline{\theta}) [(\partial_{\theta})_A \triangleright g(\underline{\theta})] d_L^n \theta & = \int [f(\underline{\theta}) \triangleleft (\partial_{\theta})_A] g(\underline{\theta}) d_L^n \theta, \\
\int d_R^n \theta [f(\underline{\theta}) \triangleleft (\partial_{\theta})_A] g(\underline{\theta}) & = \int d_R^n \theta f(\underline{\theta}) [(\partial_{\theta})_A \triangleright g(\underline{\theta})].
\end{aligned} \tag{168}$$

For this to verify, one has to take into account that

$$\begin{aligned} f(\partial \triangleright g) &= \partial_{(2)} \triangleright [(f \triangleleft \partial_{(1)})g], \\ (f \triangleleft \partial)g &= [f(\partial_{(1)} \triangleright g)] \triangleleft \partial_{(2)}. \end{aligned} \quad (169)$$

Furthermore, it should be stressed that translational invariance of our Grassmann integrals can alternatively be expressed as

$$\begin{aligned} (1 \otimes \int \cdot d_L^n \theta) \circ \Delta_{\bar{R}} f(\underline{\theta}) &= (\int \cdot d_L^n \theta \otimes 1) \circ \Delta_{\bar{R}} f(\underline{\theta}) \\ &= \int f(\underline{\theta}) d_L^n \theta, \end{aligned} \quad (170)$$

$$\begin{aligned} (1 \otimes \int d_R^n \theta \cdot) \circ \Delta_L f(\underline{\theta}) &= (\int d_R^n \theta \cdot \otimes 1) \circ \Delta_L f(\underline{\theta}) \\ &= \int d_R^n \theta f(\underline{\theta}). \end{aligned} \quad (171)$$

The above statements can be proved in a straightforward manner by insertion of the explicit expressions for superintegral and coproduct. Let us also notice that in Ref. [40] this property was taken as abstract definition for an integral on quantum spaces. In our case, integrals are given by explicit instructions being compatible with the requirement of translational invariance.

Next, let us make contact with q-analogs of δ -functions. For a δ -function on q-deformed Grassmann algebras we require to hold:

$$\begin{aligned} \int f(\underline{\theta}) \delta_{L/L}^n(\underline{\theta}) d_{L/L}^n \theta &= \int \delta_{L/L}^n(\underline{\theta}) f(\underline{\theta}) d_{L/L}^n \theta = f', \\ \int d_{R/R}^n \theta f(\underline{\theta}) \delta_{R/R}^n(\underline{\theta}) &= \int d_{R/R}^n \theta \delta_{R/R}^n(\underline{\theta}) f(\underline{\theta}) = f'. \end{aligned} \quad (172)$$

It is not very difficult to show that these requirements are satisfied by

a) (quantum plane)

$$\begin{aligned} \delta_L^2(\underline{\theta}) &= \delta_R^2(\underline{\theta}) = \theta^2 \theta^1, \\ \delta_{\bar{L}}^2(\underline{\theta}) &= \delta_{\bar{R}}^2(\underline{\theta}) = \theta^1 \theta^2, \end{aligned} \quad (173)$$

b) (three-dimensional Euclidean space)

$$\begin{aligned} \delta_L^3(\underline{\theta}) &= \delta_R^3(\underline{\theta}) = \theta^+ \theta^3 \theta^-, \\ \delta_{\bar{L}}^3(\underline{\theta}) &= \delta_{\bar{R}}^3(\underline{\theta}) = \theta^- \theta^3 \theta^+, \end{aligned} \quad (174)$$

c) (four-dimensional Euclidean space)

$$\begin{aligned}\delta_L^4(\underline{\theta}) &= \delta_R^4(\underline{\theta}) = \theta^4 \theta^3 \theta^2 \theta^1, \\ \delta_L^4(\underline{\theta}) &= \delta_R^4(\underline{\theta}) = \theta^1 \theta^2 \theta^3 \theta^4,\end{aligned}\tag{175}$$

d) (q-deformed Minkowski space)

$$\begin{aligned}\delta_L^4(\underline{\theta}) &= \theta^- \theta^{3/0} \theta^3 \theta^+, & \delta_R^4(\underline{\theta}) &= \theta^+ \theta^3 \theta^{3/0} \theta^-, \\ \delta_L^4(\underline{\theta}) &= \theta^+ \theta^{3/0} \theta^3 \theta^-, & \delta_R^4(\underline{\theta}) &= \theta^- \theta^3 \theta^{3/0} \theta^+.\end{aligned}\tag{176}$$

Last but not least we would like to say a few words about the connection between q-deformed superexponentials and translations of q-deformed supernumbers. That translations on quantum spaces are indeed given by the coproduct can also be seen from the existence of some sort of q-deformed Taylor rules for which we have [44]

$$\begin{aligned}f(\psi \oplus_L \theta) &= \exp(\psi_R \mid (\hat{\partial}_\theta)_L) \bar{\triangleright} f(\underline{\theta}), \\ f(\psi \oplus_{\bar{L}} \theta) &= \exp(\psi_{\bar{R}} \mid (\partial_\theta)_L) \triangleright f(\underline{\theta}),\end{aligned}\tag{177}$$

$$\begin{aligned}f(\theta \oplus_{\bar{R}} \psi) &= f(\underline{\theta}) \triangleleft \exp((\hat{\partial}_\theta)_R \mid \psi_{\bar{L}}), \\ f(\theta \oplus_R \psi) &= f(\underline{\theta}) \bar{\triangleleft} \exp((\partial_\theta)_{\bar{R}} \mid \psi_L).\end{aligned}\tag{178}$$

Again, these identities can be verified in a straightforward manner making use of the explicit form for the superexponentials and the action of derivatives on antisymmetrized quantum spaces.

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A Quantum spaces

In this appendix we list for the quantum spaces under consideration the explicit form of their defining relations and the non-vanishing elements of their quantum metrics.

The coordinates of two-dimensional antisymmetrized Manin plane fulfill the relation [38, 47]

$$\theta^1\theta^2 = -q^{-1}\theta^2\theta^1. \quad (179)$$

The q -deformed spinor metric is given by a matrix ε^{ij} with non-vanishing elements

$$\varepsilon^{12} = q^{-1/2}, \quad \varepsilon^{21} = -q^{1/2}. \quad (180)$$

Indices can be raised and lowered as usual, i.e.

$$\theta^\alpha = \varepsilon^{\alpha\beta}\theta_\beta, \quad \theta_\alpha = \varepsilon_{\alpha\beta}\theta^\beta, \quad (181)$$

where $\varepsilon_{ij} = -\varepsilon^{ij}$.

The commutation relations defining an antisymmetrized version of three-dimensional q -deformed Euclidean space read

$$\begin{aligned} (\theta^+)^2 &= (\theta^-)^2 = 0, \\ (\theta^3)^2 &= \lambda\theta^+\theta^-, \\ \theta^+\theta^- &= -\theta^-\theta^+, \\ \theta^\pm\theta^3 &= -q^{\pm 2}\theta^3\theta^\pm. \end{aligned} \quad (182)$$

The non-vanishing elements of the corresponding quantum metric are

$$g^{+-} = -q, \quad g^{33} = 1, \quad g^{-+} = -q^{-1}. \quad (183)$$

Covariant coordinates can be introduced by

$$\theta_A = g_{AB}\theta^B, \quad (184)$$

with g_{AB} being the inverse of g^{AB} .

For antisymmetrized q -deformed Euclidean space with four dimensions we have the relations

$$\begin{aligned} (\theta^i)^2 &= 0, \quad i = 1, \dots, 4, \\ \theta^1\theta^2 &= -q^{-1}\theta^2\theta^1, \\ \theta^1\theta^3 &= -q^{-1}\theta^3\theta^1, \\ \theta^2\theta^4 &= -q^{-1}\theta^4\theta^2, \\ \theta^3\theta^4 &= -q^{-1}\theta^4\theta^3, \\ \theta^1\theta^4 &= -\theta^4\theta^1, \\ \theta^2\theta^3 &= -\theta^3\theta^2 + \lambda\theta^1\theta^4, \end{aligned} \quad (185)$$

and its metric has the non-vanishing components

$$g^{14} = q^{-1}, \quad g^{23} = g^{32} = 1, \quad g^{41} = q. \quad (186)$$

The generators of antisymmetrized q -deformed Minkowski space [16, 17, 21] are subject to the relations

$$\begin{aligned} (\theta^\mu)^2 &= 0, \quad \mu \in \{+, -, 0\}, \\ \theta^3 \theta^\pm &= -q^{\mp 2} \theta^\pm \theta^3, \\ \theta^3 \theta^3 &= \lambda \theta^+ \theta^-, \\ \theta^+ \theta^- &= -\theta^- \theta^+, \\ \theta^\pm \theta^0 + \theta^0 \theta^\pm &= \pm q^{\mp 1} \lambda \theta^\pm \theta^3, \\ \theta^0 \theta^3 + \theta^3 \theta^0 &= \lambda \theta^+ \theta^-. \end{aligned} \quad (187)$$

Instead of dealing with the coordinate θ^3 or θ^0 it is often more convenient to work with the light-cone coordinate $\theta^{3/0} = \theta^3 - \theta^0$, for which we have the additional relations

$$\begin{aligned} (\theta^{3/0})^2 &= 0, \\ \theta^\pm \theta^{3/0} &= -\theta^{3/0} \theta^\pm, \\ \theta^0 \theta^{3/0} + \theta^{3/0} \theta^0 &= -\lambda \theta^+ \theta^-, \\ \theta^\pm \theta^0 + q^{\pm 2} \theta^0 \theta^\pm &= \pm q^{\pm 1} \lambda \theta^\pm \theta^{3/0}, \\ \theta^3 \theta^{3/0} + \theta^{3/0} \theta^3 &= -\lambda \theta^+ \theta^-. \end{aligned} \quad (188)$$

Finally, we write down the non-vanishing entries of the matrix representing q -deformed Minkowski metric:

$$\eta^{00} = -1, \quad \eta^{33} = 1, \quad \eta^{+-} = -q, \quad \eta^{-+} = -q^{-1}. \quad (189)$$

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